Student handout

The Electrostatic Potential Due to a Ring of Charge

1. Use the superposition principle for the electrostatic potential due to a continuous charge distribution:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau', \tag{1}$$

to find the electrostatic potential everywhere in space due to a uniformly charged ring with radius R and total charge Q.

Check with a teaching team member before moving on to subsequent parts below.

2. Evaluate your expression for the special case of the potential on the z-axis.
3. Evaluate your expression for the special case of the potential on the x-axis.
4. Find a series expansion for the electrostatic potential in these special regions:
   a) Near the center of the ring, in the plane of the ring;
   b) Near the center of the ring, on the axis of the ring;
   c) Far from the ring on the axis of symmetry;
   d) Far from the ring, in the plane of the ring.

1 Instructor's Guide

1.1 Introduction

Part I - Finding the potential everywhere in space

Students should be assigned to work in small groups given the following instructions using the visual of a hula hoop or other large ring:

Prompt: "This is a ring with radius R and total charge Q. Find a formula for the electrostatic potential V due to this ring that is valid everywhere in space".

Part II (Optional) - Power series expansion along an axis

With the charged ring in the x, y-plane, student groups are asked to make the power series expansion for either near or far from the plane on the z axis or near or far from the z axis in the x, y-plane. The instructor may choose to have the whole class do one particular case or have different groups do different cases, in a Compare and Contrast strategy (Compare and Contrast Activities).

1.2 Student Conversations

Part I - Finding the potential everywhere in space

- Students frequently leave math classes understanding integration primarily as "the area under a curve". This activity pushes students to generalize their understanding of integration to focus on "chop" (the region of space into small pieces), "multiply" (the rest of the integrand by a differential that represents the small chopped piece of space), and "add" (the contributions from each chopped piece). Watch for groups that don't realize that "integrate" in this sense means "find the total amount of ...".

- Watch for groups that need help recognizing that the "multiply" step yields a small amount of whatever it is that they are trying to calculate a total amount of. Some students will integrate just the charge density, without the factor of $\frac{1}{|\vec{r}-\vec{r}'|}$. Emphasize that the superposition principle says to add up potentials, not charges.

- The integral that students are trying to evaluate is an elliptic integral. Most commonly students have never seen such "unsolvable" integrals in their calculus classes and will be surprised to be asked to find a definite integral that they cannot evaluate in terms of the functions that they already know. Let them try to evaluate the integral briefly, but not so long tha they get frustrated. Eventually, explain what is happening and tell them that they should stop when they have an expression that a computer algebra system like *Mathematica* could evaluate numerically to find the potential at a given point, i.e.

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi R}\int_0^{2\pi}\frac{1}{\sqrt{r^2+r'^2-2rr'\cos(\phi-\phi')+z^2}}\,Rd\phi' \qquad (2)$$

  Knowing how to recognize when to stop is an important learning goal of the activity.

- The fact that the integral cannot be evaluated is a great opportunity for a mini-lecture transition about why to do power series expansions. If students do the power series expansion in the integrand, it is then it is always possible to do the integration term-by-term, (see Part II, below).

- Some students will need a few minutes to realize that the charge density is given by the total charge divided by the circumference of the ring $\lambda = \frac{Q}{2\pi R}$. Watch for those students who try only dimensional arguments (who will not get the factor of 2π), those who chant "density is charge per distance" but don't know what to do with those words, or those who try to use a formula that charge density is the derivative of charge (who will not make progress at all since they don't have a formula for charge to differentiate).

- Students must use an appropriate coordinate system to take advantage of the symmetry of the problem. Students attempting to do the problem in rectangular coordinates can be given a few minutes to struggle and see the problems that arise and then, if necessary, guided to using curvilinear coordinates. Most students will choose to do this problem in cylindrical coordinates and our solutions are written in this coordinate system, but spherical coordinate work equally well. It's probably a good idea to praise groups who choose spherical coordinates for making a reasonable choice but then ask them to switch to cylindrical so that they will be able to compare their answers to other groups and to the solutions. If you have time, having some groups do cylindrical and some do spherical can make a great compare and contrast activity, but it will add significantly to the class time needed for the activity.

- Students often mix up the primed and unprimed variables, so it is important to ask them to clarify their notation (e.g. what variable are you integrating over? which $\vec{r}$ is this? which direction does $\vec{r} - \vec{r}'$ point?, etc.) The convention we use is that $\vec{r}$ points from the origin to the point where the field is being evaluated and $\vec{r}'$ points from the origin to the source. If students argue that they have their own private convention, it might be an appropriate time for a (polite) mini-sermon on the value of a shared notation in a collaborative field like physics and especially in a collaborative learning situation like an active engagement classroom.

- Students may reach a correct figure (chopped pieces, an origin, and labeled position vectors) on their own in a few minutes or they may need help. In particular, the may miss the fact that this is a three-dimensional problem. Either a hoop or a ring drawn on the table can be used to ask students about the potential at points in space that are outside the plane of the ring.

Part II (Optional) - Power series expansion along an axis

- If you are doing this activity without having had students first create power series expansions for the electrostatic potential due to two charges, students will probably find this portion of the activity very challenging. If they have already done the Electrostatic potential due to two points Electrostatic Potential Due to a Pair of Charges (with Series) activity, or similar activity, students will probably be successful with the z axis case without a lot of assistance because it is very similar to the y axis case for the two $+Q$ point charges. However, the y axis here presents a new challenge because the "something small" is the sum of two terms. It will probably not be obvious for students to let $\epsilon = \frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}$ and suggestions should be given to avoid having them stuck for a long period of time. Once this has been done, students may also have trouble undoing the substitution and combining terms of the same order. For example the ϵ^2 term results in a both a third and a fourth order term in the expansion. Similarly, students may not realize that to get a valid third order expansion they need to include pieces of the ϵ^3 term.

1.3 Wrap-up

- Emphasize that they have learned how to "unpack" the iconic expression

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

first to understand that the r in the denominator means the distance between the charge and the point where the potential is being evaluated

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{|\vec{r} - \vec{r}'|},$$

next to chop up a continuous source into small pieces $\rho(\vec{r}')\,d\tau'$ and use the superposition principle to "sum up" the contribution to the potential from each piece

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|} \tag{3}$$

$$\Rightarrow \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau', \tag{4}$$

and finally to adjust the formula to indicate whether the charge is distributed along a one-dimensional curve $\lambda(\vec{r}')\,|d\vec{r}'|$ (this case) or a two-dimensional surface $\sigma(\vec{r}')\,dA'$ or in a three-dimensional volume $\rho(\vec{r}')\,d\tau'$.

$$V(\vec{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\lambda(\vec{r}')}{|\vec{r}-\vec{r}'|}\,|d\vec{r}'| & \text{one-dimensional source} \\ \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dA' & \text{two-dimensional source} \\ \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau' & \text{three-dimensional source} \end{cases}.$$

- Emphasize that while one may not be able to perform a particular integral, the power series expansion of that integrand can always be integrated *term by term* because we know how to integrate powers.

- Discuss which quantities are variable and which variables are held constant. Students frequently think of anything represented by a letter as a "variable" and do not realize that for each particular situation certain quantities remain constant during integration. For example students frequently do not see that the R representing the radius of the ring is held constant during the integration over all space while the r representing the distance to the origin is varying. Understanding this is something professional physicists do automatically while students frequently don't even consider it. This is an important discussion that helps students understand this particular ring problem and also lays the groundwork for better understanding of integration in a variety of contexts.

- It is very helpful to end this activity with a way to visualize the value of the potential everywhere in space. *Maple/Mathematica* representation of elliptic integral - After finding the elliptic integral and doing the power series expansion, students can see what electric potential "looks like" over all space by using the activities:guides:vfvring.nb—Mathematica worksheet.