Student handout

Eigenvalues and Eigenvectors

Each group will be assigned one of the following matrices.

$$A_1 \doteq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad A_2 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_3 \doteq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_4 \doteq \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \qquad A_5 \doteq \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix} \qquad A_6 \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_7 \doteq \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$A_8 \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A_9 \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For your matrix:

1. Find the eigenvalues.
2. Find the (unnormalized) eigenvectors.
3. Describe what this transformation does.
4. Normalize your eigenvectors. (Quantum eigenstates are represented by normalized eigenvectors.)

If you finish early, try another matrix with a different structure, *i.e.* real vs. complex entries, diagonal vs. non-diagonal, 2×2 vs. 3×3, with vs. without explicit dimensions.

1 Instructor's Guide

1.1 Main Ideas

This is a small group activity for groups of 3-4. The students will be given one of 10 matrices. The students are then instructed to find the eigenvectors and eigenvalues for this matrix and record their calculations on their medium-sized whiteboards. In the class discussion that follows students report their finding and compare and contrast the properties of the eigenvalues and eigenvectors they find.

Two topics that should specifically discussed are the case of repeated eigenvalues (degeneracy) and complex eigenvectors, e.g., in the case of some pure rotations, special properties of the eigenvectors and eigenvalues of hermitian matrices, common eigenvectors of commuting operators.

1.2 Students' Task

1.3 Introduction

Give a mini-lecture on how to calculate eigenvalues and eigenvectors. It is often easiest to do this with an example. We like to use the matrix

$$A_7 \doteq \begin{pmatrix} 1 & 2 \\ 9 & 4 \end{pmatrix}$$

from the `https://paradigms.oregonstate.edu/activities/2179` `https://paradigms.oregonstate.edu/activities/2179` Finding Eigenvectors and Eigenvalues since the students have already seen this matrix and know what it's eigenvectors are. Then every group is given a handout, assigned a matrix, and then asked to: - Find the eigenvalues - Find the (unnormalized) eigenvectors - Normalize the eigenvectors - Describe what this transformation does

1.4 Student Conversations

- Typically, students can find the eigenvalues without too much problem. Eigenvectors are a different story. To find the eigenvectors, they will have two equations with two unknowns. They expect to be able to find a unique solution. But, since any scalar multiple of an eigenvector is also an eigenvector, their two equations will be redundant. Typically, they must choose any convenient value for one of the components (e.g. $x = 1$) and solve for the other one. Later, they can use this scale freedom to normalize their vector.

- The examples in this activity were chosen to include many of the special cases that can trip students up. A common example is when the two equations for the eigenvector amount to something like $x = x$ and $y = -y$. For the first equation, they may need help to realize that $x =$ "anything" is the solution. And for the second equation, sadly, many students need to be helped to the realization that the only solution is $y = 0$.

1.5 Wrap-up

The majority of the this activity is in the wrap-up conversation.

The [[whitepapers:narratives:eigenvectorslong|Eigenvalues and Eigenvectors Narrative]] provides a detailed narrative interpretation of this activity, focusing on the wrap-up conversation.

- **Complex eigenvectors:** connect to discussion of rotations in the Linear Transformations activity where there did not appear to be any vectors that stayed the same.

- **Degeneracy:** Define degeneracy as the case when there are repeated eigenvalues. Make sure the students see that, in the case of degeneracy, an entire subspace of vectors are all eigenvectors.

- **Diagonal Matrices:** Discuss that diagonal matrices are trivial. Their eigenvalues are just their diagonal elements and their eigenvectors are just the standard basis.
- **Matrices with dimensions:** Students should see from these examples that when you multiply a transformation by a real scalar, its eigenvalues are multiplied by that scalar and its eigenvectors are unchanges. If the scalar has dimensions (e.g. $\hbar/2$), then the eigenvalues have the same dimensions.