Student handout

Linear Transformations

1. Using colored markers, draw these initial vectors, all on the same graph on your whiteboard.

$$\vec{v}_{red} = 1|\hat{x}\rangle + 0|\hat{y}\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{v}_{green} = 0|\hat{x}\rangle + 1|\hat{y}\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{v}_{blue} = 1|\hat{x}\rangle + 1|\hat{y}\rangle \doteq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_{black} = 1|\hat{x}\rangle - 1|\hat{y}\rangle \doteq \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \vec{v}_{purple} = 1|\hat{x}\rangle + 3|\hat{y}\rangle \doteq \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2. Each group will be assigned one of the following matrices. Operate on the initial vectors with your group's matrix and graph the transformed vectors on a single (new) graph.

$$A_1 \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad A_2 \doteq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A_3 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A_4 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_5 \doteq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad A_6 \doteq \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad A_7 \doteq \begin{pmatrix} 1 & 2 \\ 9 & 4 \end{pmatrix} \quad A_8 \doteq \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A_9 \doteq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad A_{10} \doteq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad A_{11} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A_{12} \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. Find the determinant of your matrix.

4. Make note of any differences between the initial and transformed vectors. Specifically, look for rotations, inversions, length changes, anything that is different. Are there any vectors which are left unchanged by your transformation? Your group should be prepared to report to the class about your transformation.

5. After all groups are done, you will be reporting to the class:

   - What is your matrix (give the number and read the elements from top left to bottom right)
   - What is the determinant of your matrix?
   - What does your matrix do (geometrically) to vectors?
   - Are there any vectors that don't change direction?

6. If you get done early, save your original board for the report and work on some other examples that are structurally different from your original example.

1 Instructor's Guide

1.1 Students' Task

Small groups of students are given a list of vectors to draw in different colors, and each group is assigned one of 10 transformation matrices. The students are then instructed to operate on the vectors with the matrix, and observe the changes in the vectors.

The class discussion that follows focuses on the changes caused by the different matrices and the class as a whole proposes hypotheses about the geometric meaning of the determinant of the matrix. Finally, the discussion is brought round to the topic of the vectors that are unchanged by the matrix, and the students are correctly led to identify those vectors as the eigenvectors of the matrix.

1.2 Student Conversations

The activity itself is quite trivial. Students simply need to do a little matrix multiplication and plotting of vectors. (Although for weak students, working out the relationship between the algebraic representation of the vectors and their geometric plots can be quite valuable.)

1.3 Wrap-up

The [[whitepapers:narratives:lineartranslong|Linear Transformations Narrative]] provides a detailed narrative interpretation of this activity, focusing on the wrap-up conversation.

The real power of this activity comes from the wrap-up. As groups finish, they should be asked to record the following information on boards around the room: their matrix, their transformed vectors, as short description of what the transformation does to the vectors, the determinant of their matrix, and a list of any vectors that are unchanged (or whose direction is unchanged) by the transformation. If a group finishes early, they should be directed to do another example.

When the groups are done, each group is asked to give a mini-report listing the information they put on the board. The class as a whole is directed to "hypothesize" about the geometric meaning of the determinant. As each successive example is considered, the running class hypothesis(es) is refined to consider the new information. We find it useful to discuss with the students the relationship of this type of problem-solving to the discovery process in science. FIXME (We should add a videoclip of 2007?)

At the end of the activity, if students have not spontaneously asked about the relationship of this activity to eigenvectors and eigenvalues, they should be asked about this explicitly. (In our course, some, but not all students have had a course in linear algebra.) They should be led to the definition that eigenvectors are the vectors whose direction is unchanged by the transformation and eigenvalues are the amount of stretch that the eigenvectors experience. A discussion of the fact that to scientists and mathematicians, multiplication by a negative scalar does not change the "direction" of a vector differs from the everyday speech convention that "north" and "south" are different directions.

2 Extensions

We refer back to this activity often, especially during a later activity Finding Eigenvectors and Eigenvalues