1 Instructor's Guide

This activity is part of the Arms Sequence for Complex Numbers and Quantum States.

1.1 Prerequisite Ideas

Students should be familiar with complex numbers, particularly Argand diagrams and rectangular, polar, and exponential forms for complex numbers.

Students should also have been introduced to the idea that quantum states are represented by complex-valued vectors.

Students should be familiar with the spin basis kets: $\left|+\right\rangle_z$, $\left|-\right\rangle_z$, $\left|+\right\rangle_x$, $\left|-\right\rangle_x$, $\left|+\right\rangle_y$, and $\left|-\right\rangle_y$, written in the S_z basis in both matrix notation and Dirac Notation.

1.2 Prompts:

1. **Set-Up**

   - Students should form pairs to represent a quantum state.
   - Each student represents one of the S_z basis states:

     person on the left $\rightarrow \left|+\right\rangle$
     person on the right $\rightarrow \left|-\right\rangle$

   - Each student uses their left arm to represent the complex probability amplitude for their basis ket.

2. **Optional Warm-Up** Have each student represent a complex number given in rectangular, polar, or exponential form.

3. "Represent the state: spin up in the $z-$direction." Write the $\left|+\right\rangle_z$ ket on the board.

   - The student on the left should be pointing straight forward.
   - The student on the right should have zero length. Encourage students to bring them hand in to their shoulder - leaving the arm dangling is ambiguous with pointing along the negative imaginary axis.
   - Ask "What does this state look like in matrix representation?"

4. "Represent the state: spin down in the $z-$direction." Write the $\left|-\right\rangle_z$ ket on the board.

   - Now, the student on the left should have zero length and the student on the right should be pointing straight forward.
   - **Be on the Lookout:** Some incorrect answer to expect include:

- Left person's arm pointing backward. Some students thing of spin up and spin down as pointing in opposite directions.
- Left person's arm straight up (pure imaginary). Some students key in on the orthogonality of $|+\rangle_z$ and $|-\rangle_z$ and think the arms need to be perpendicular.
- Right person's arm straight up (pure imaginary). (This is a technically correct answer, but with a different overall phase. It's not worth getting into it at the moment.) Some students key in on the orthogonality of $|+\rangle_z$ and $|-\rangle_z$ and think the arms need to be perpendicular.

- **Optional:** "Are these two states spin up and spin down in the $z-$direction orthogonal to each other?".
  - Some students will be confused because the arms are not perpendicular to each other.
  - They are orthogonal because their inner product is zero. Orthogonal vectors don't look perpendicular in the Arms representation.
- Ask "What does this state look like in matrix representation?"

5. "Represent the state: spin up in the $x-$direction." Write the $|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z$ ket on the board.
   - The two arms should be parallel (and pointing forward).
   - **Be on the Lookout:** Some students will now bend their arms to accommodate the smaller norm of the complex number. Encourage students to keep their arms straight. This is will important so that they can better see the relative angle between arms. The Arms representation is not good at representing the norm of a complex number, but is good for examining the relative angles between arms.
   - Ask "What does this state look like in matrix representation?"

6. "Represent the state: spin down in the $x-$direction." Write the $|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle_z - \frac{1}{\sqrt{2}}|-\rangle_z$ ket on the board.
   - The two arms should be antiparallel.
   - **Optional:** "Are these two states spin up and spin down in the x-direction orthogonal to each other?".
     - They are orthogonal because their inner product is zero. Orthogonal vectors don't look perpendicular in the Arms representation.
   - **Be on the Lookout:** At some stage now, the students will realize that because of the choice to make the first coefficient real and positive, the person on the left will always point forward (or be zero).

7. "Represent the state: spin up in the $y-$direction." Write the $|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{i}{\sqrt{2}}|-\rangle_z$ ket on the board.
   - The two arms should be perpendicular.

- Ask "What does this state look like in matrix representation?"

8. "Represent the state: spin down in the $y-$direction." Write the $\left|-\right\rangle_y = \frac{1}{\sqrt{2}}\left|+\right\rangle_z - \frac{i}{\sqrt{2}}\left|-\right\rangle_z$ ket on the board.

   - The two arms should be perpendicular.
   - **Optional:** "Are these two states spin up and spin down in the y-direction orthogonal to each other?".
     - They are orthogonal because their inner product is zero. Orthogonal vectors don't look perpendicular in the Arms representation.
     - **Be on the Lookout:** Students will notice that for both $\left|\pm\right\rangle_x$ and $\left|\pm\right\rangle_x$, they are orthogonal to each other and with arms they are different by 180 degrees for the person on the right. As long as neither of the coefficients is zero, the relative phases two orthongal vectors should be different by π:

$$\left|\psi_1\right\rangle = a\left|+\right\rangle + be^{i\theta}\left|-\right\rangle$$
$$\left|\psi_2\right\rangle = c\left|+\right\rangle + de^{i\alpha}\left|-\right\rangle$$
$$ac^* = -bd^*e^{i(\theta-\alpha)}$$

Since a, b, c, d are real and positive in this form (and assuming they are non-zero):

$$ac = -bde^{i(\theta-\alpha)}$$
$$\rightarrow e^{i(\theta-\alpha)} = -1$$
$$\theta - \alpha = \pi$$

     - Ask "What does this state look like in matrix representation?"

1.3 Wrap-up

Little wrap-up is needed. Review points about:

- The Arms representation represents the complex components of a vector.
- To represent a two-state quantum system, you need two arms.
- The Arms representation is similar to matrix representation in the the basis is not represented explicitly - you just have to know what it is.
- The Arms representation doesn't represent the norm of complex numbers well, but does represent the phase angle well.
- **Optional:** Orthogonal states don't look spatially perpendicular.
- **Optional foreshadowing:** Being able to recognize the Arms representation of the standard spin basis vectors will be useful later.