Student handout

Outer Product of a Vector on Itself

1. Your group will be given a pair (or triple) of vectors below, find the matrix that is the outer product of each vector on itself (i.e., $|v_1\rangle\langle v_1|$)? All the vectors are written in the S_z basis.

1) $|+\rangle \doteq \begin{bmatrix}1\\0\end{bmatrix}$ $\quad |-\rangle \doteq \begin{bmatrix}0\\1\end{bmatrix}$

2) $|+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad |-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

3) $|+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ $\quad |-\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$

4) $|v_7\rangle \doteq \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}$ $\quad |v_8\rangle \doteq \frac{1}{5}\begin{bmatrix}4\\-3\end{bmatrix}$

5) $|v_9\rangle \doteq \begin{bmatrix}a\\be^{i\phi}\end{bmatrix}$ $\quad |v_{10}\rangle \doteq \begin{bmatrix}b\\-ae^{i\phi}\end{bmatrix}$

6) $|1\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\1\\\frac{1}{\sqrt{2}}\end{bmatrix}$ $\quad |0\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\quad |-1\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\-1\\\frac{1}{\sqrt{2}}\end{bmatrix}$

2. What is the square of each of your outer products?
3. What is the product of each pair of your outer products?
4. For each row of vectors, add all of the outer products.
5. What is the determinant of each of your outer products?
6. What is the transformation caused by each of your outer products?

   Bonus: How would you answer questions (2), (3), (4) staying purely in Dirac bra-ket notation?

1 Instructor's Guide

1.1 Prompt:

I recommend doing the activity as a Compare & Contrast. Make sure students who get the first two do a second one. Have groups record their results on a public table.

1.2 Student Conservations

- Students may have trouble identifying the transformation.
  - For real vectors, have them plot the vectors on the $|\pm\rangle$ axes.
  - For imaginary vectors, try factoring out a common factor from both components.
- Students will be curious about whether the matrices are projections or scrinches/smooshes. You can point out that the scaling factor on their transformed vector is the inner product between their original vector and the untransformed vector: $(|v_i\rangle \langle v_i|) |\psi\rangle = \langle v_i | \psi\rangle |v_i\rangle = |\psi'\rangle$

1.3 Activity: Wrap-up

This activity works well if different groups are assigned different vectors and the different results are reported at the end. Wrap-up should emphasize that:

- an outer product of two vectors produces a matrix
- an outer product of a vector $|a\rangle$ on itself produces an operator that projects vectors onto the line with the same slope as the $|a\rangle$.
- the determinant of a projection operator is zero
- A projection operator squares to itself

If students have done the `https://paradigms.oregonstate.edu/act/2221`, the facilitator can point out that a projection (and renormalization) operation is consistent with the transformation that occurs when a Stern-Gerlach measurement is made.

This activity works well as a follow-up to the Linear Transformations activity.