Lecture about finding $|\pm\rangle_x$ and then $|\pm\rangle_y$. There are two conventional choices to make: relative phase for ${}_x\langle+|-\rangle_x$ and ${}_y\langle+|+\rangle_x$.

So far, we've talked about how to calculate measurement probabilities if you know the input and output quantum states using the probability postulate:

$$\mathcal{P} = |\langle\psi_{out}|\psi_{in}\rangle|^2$$

Now we're going to do this process in reverse.

I want to be able to relate the output states of Stern-Gerlach analyzers oriented in different directions to each other (like $|\pm\rangle_x$ and $|\pm\rangle_x$ to $|\pm\rangle$). Since $|\pm\rangle$ forms a basis, I can write any state for a spin-1/2 system as a linear combination of those states, including these special states.

I'll start with $|+\rangle_x$ written in the S_z basis with general coefficients:

$$|+\rangle_x = a|+\rangle + be^{i\phi}|-\rangle$$

Notice that:

(1) a, b, and ϕ are all real numbers; (2) the relative phase is loaded onto the second coefficient only.

My job is to use measurement probabilities to determine a, b, and ϕ.

I'll prepare a state $|+\rangle_x$ and then send it through x, y, and z analyzers. When I do that, I see the following probabilities:

Input = $\vert+\rangle_x$	S_x	S_y	S_z
$P(\hbar/2)$	1	1/2	1/2
$P(-\hbar/2)$	0	1/2	1/2

First, looking at the probability for the S_z components:

$$(S_z = +\hbar/2) = |\langle+|+\rangle_x|^2 = 1/2$$

Plugging in the $|+\rangle_x$ written in the S_z basis:

$$1/2 = \left|\langle+|\Big(a|+\rangle + be^{i\phi}|-\rangle\Big)\right|^2$$

Distributing the $\langle+|$ through the parentheses and use orthonormality:

$$\begin{aligned} 1/2 &= \left|a\cancelto{1}{\langle+|+\rangle} + be^{i\phi}\cancelto{0}{\langle+|-\rangle}\right|^2 \\ &= |a|^2 \end{aligned}$$

$$\rightarrow a = \frac{1}{\sqrt{2}}$$

Similarly, looking at $S_z = -\hbar/2$:

$$(S_z = +\hbar/2) = |\langle -|+\rangle_x|^2 = 1/2$$

$$1/2 = \left|\langle -|\left(a|+\rangle + be^{i\phi}|-\rangle\right)\right|^2$$

$$1/2 = \left|a\cancelto{0}{\langle -|+\rangle} + be^{i\phi}\cancelto{1}{\langle -|-\rangle}\right|^2$$

$$= |be^{i\phi}|^2$$

$$= |b|^2\cancelto{1}{(e^{i\phi})(e^{-i\phi})}$$

$$\rightarrow b = \frac{1}{\sqrt{2}}$$

I can't yet solve for ϕ but I can do similar calculations for $|-\rangle_x$:

Input = $|-\rangle_x$	S_x	S_y	S_z
$P(\hbar/2)$	0	1/2	1/2
$P(-\hbar/2)$	1	1/2	1/2

$$|-\rangle_x = c|+\rangle + de^{i\gamma}|-\rangle$$

$$(S_z = +\hbar/2) = |\langle +|-\rangle_x|^2 = 1/2$$

$$\rightarrow c = \frac{1}{\sqrt{2}}$$

$$(S_z = +\hbar/2) = |\langle -|-\rangle_x|^2 = 1/2$$

$$\rightarrow d = \frac{1}{\sqrt{2}}$$

So now I have:

$$|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\beta}|-\rangle$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\gamma}|-\rangle$$

I know $\beta \neq \gamma$ because these are not the same state - they are orthogonal to each other:

$$0 = {}_x\langle +|-\rangle_x$$

$$= \left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}e^{i\beta}\langle -|\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\gamma}|-\rangle\right)$$

Now FOIL like mad and use orthonormality:

$$\begin{aligned} 0 &= \frac{1}{2}\left(\cancelto{1}{\langle+|+\rangle} + e^{i\gamma}\cancelto{0}{\langle+|-\rangle} + e^{i\beta}\cancelto{0}{\langle-|+\rangle} + e^{i(\gamma-\beta)}\cancelto{1}{\langle-|-\rangle}\right) \\ &= \frac{1}{2}\left(1 + e^{i(\gamma-\beta}\right) \\ \rightarrow \quad & e^{i(\gamma-\beta)} = -1 \end{aligned}$$

This means that $\gamma - \beta = \pi$. I don't have enough information to solve for β and γ, but there is a one-time conventional choice made that $\beta = 0$ and $\gamma = 1$, so that:

$$\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}\cancelto{1}{e^{i0}}|-\rangle \\ |-\rangle_x &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}\cancelto{-1}{e^{i\pi}}|-\rangle \end{aligned}$$

$$\begin{aligned} \rightarrow |+\rangle_x &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \\ |-\rangle_x &= \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle \end{aligned}$$

When $|\pm\rangle_y$ is the input state:

Input = $|+\rangle_y$	S_x	S_y	S_z
$P(\hbar/2)$	1/2	1	1/2
$P(-\hbar/2)$	1/2	0	1/2

Input = $|-\rangle_y$	S_x	S_y	S_z
$P(\hbar/2)$	1/2	0	1/2
$P(-\hbar/2)$	1/2	1	1/2

The calculations proceed in the same way. The S_z probabilities give me:

$$\begin{aligned} |+\rangle_y &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}\cancelto{1}{e^{i\alpha}}|-\rangle \\ |-\rangle_y &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}\cancelto{-1}{e^{i\theta}}|-\rangle \end{aligned}$$

The orthongality between $|+\rangle_y$ and $|-\rangle_y$ mean that $\theta - \alpha = \pi$.

But I also know the S_x probabilities and how to write $|ket\pm_x$ in the S_z basis. For an input of $|+\rangle_y$:

$$\begin{aligned}
(S_x = +\hbar/2) &= |\,_x\langle+|+\rangle_y|^2 = 1/2 \\
1/2 &= \left|\left(\frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}\langle-|\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|-\rangle\right)\right|^2 \\
1/2 &= \left|\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\cancelto{1}{\langle+|+\rangle} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{i\alpha}\cancelto{1}{\langle-|-\rangle}\right|^2 \\
&= \frac{1}{4}|1+e^{i\alpha}|^2 \\
&= \frac{1}{4}\left(1+e^{i\alpha}\right)\left(1+e^{-i\alpha}\right) \\
&= \frac{1}{4}\left(2+e^{i\alpha}+e^{-i\alpha}\right) \\
&= \frac{1}{4}\left(2+2\cos\alpha\right) \\
\frac{1}{2} &= \frac{1}{2}+\frac{1}{2}\cos\alpha \\
0 &= \cos\alpha \\
\rightarrow \alpha &= \pm\frac{\pi}{2}
\end{aligned}$$

Here, again, I can't solve exactly for alpha (or θ), but the convention is to choose $alpha = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, making

$$\begin{aligned}
|+\rangle_y &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}\cancelto{i}{e^{i\pi/2}}|-\rangle \\
|-\rangle_y &= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}\cancelto{-i}{e^{i3\pi/2}}|-\rangle \\
\rightarrow |+\rangle_y &= \frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle \\
|-\rangle_y &= \frac{1}{\sqrt{2}}|+\rangle - \frac{i}{\sqrt{2}}|-\rangle
\end{aligned}$$

If I use these two convenctions for the relative phases, then I can write down $|\pm\rangle_n$ in an arbitrary direction described by the spherical coordinates θ and ϕ as:

Discuss the generalize eigenstates:

$$\begin{aligned}
|+\rangle_n &= \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle \\
|-\rangle_n &= \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}e^{i\phi}|-\rangle
\end{aligned}$$

And how the $|\pm\rangle_x$ and $|\pm\rangle_y$ are consistent.