Student handout Consider a system of n different masses m_i, interacting with each other and being acted on by external forces. We can write Newton's second law for the positions $\vec{r}_i$ of each of these masses with respect to a fixed origin $\mathcal{O}$, thereby obtaining a system of equations governing the motion of the masses.

$$\begin{aligned} m_1 \frac{d^2\,\vec{r}_1}{dt^2} &= \vec{F}_1 + \ \ 0 \ \ + \vec{f}_{12} + \vec{f}_{13} + \ \ \ldots \ \ + \vec{f}_{1n} \\ m_2 \frac{d^2\,\vec{r}_2}{dt^2} &= \vec{F}_2 + \vec{f}_{21} + \ \ 0 \ \ + \vec{f}_{23} + \ \ \ldots \ \ + \vec{f}_{2n} \\ &\vdots \\ m_n \frac{d^2\,\vec{r}_n}{dt^2} &= \vec{F}_n + \vec{f}_{n1} + \vec{f}_{n2} + \cdots + \vec{f}_{n(n-1)} + 0 \end{aligned} \tag{1}$$

Here, we have chosen the notation $\vec{F}_i$ for the net external forces acting on mass m_i and $\vec{f}_{ij}$ for the internal force of mass m_j acting on m_i.

In general, each internal force $\vec{f}_{ij}$ will depend on the positions of the particles $\vec{r}_i$ and $\vec{r}_j$ in some complicated way, making (1), a set of **coupled** differential equations. To solve (1), we first need to **decouple** the differential equations, i.e. find an equivalent set of differential equations in which each equation contains only one variable.

The weak form of Newton's third law states that the force $\vec{f}_{12}$ of m_2 on m_1 is equal and opposite to the force $\vec{f}_{21}$ of m_1 on m_2. We see that each internal force appears twice in the system of equations (1), once with a positive sign and once with a negative sign. Therefore, if we add all of the equations together, the internal forces will all cancel, leaving:

$$\sum_{i=1}^{n} m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_{i=1}^{n} \vec{F}_i \tag{2}$$

Notice what a surprising equation (2) is. The right-hand side directs us to add up all of the external forces, each of which acts on a different mass; something you were taught never to do in introductory physics.

The left-hand side of (2) directs us to add up (the second derivatives of) n "weighted" position vectors pointing from the origin to different masses. We can simplify the left-hand side of (2) if we multiply and divide by the total mass $M = m_1 + m_2 + \cdots + m_n$ and use the linearity of differentiation to "factor out" the derivative operator:

$$\sum_{i=1}^{n} m_i \frac{d^2\vec{r}_i}{dt^2} = M \frac{d^2}{dt^2} \left(\sum_{i=1}^{n} \frac{m_i}{M}\,\vec{r}_i \right) \tag{3}$$

$$= M \frac{d^2\vec{R}_{cm}}{dt^2} \tag{4}$$

We recognize (or define) the quantity in the parentheses on the right-hand side of (3) as the position vector $\vec{R}_{cm}$ from the origin to the "center of mass" of the system of particles, i.e.

$$\vec{R}_{cm} = \sum_{i=1}^{n} \frac{m_i}{M}\,\vec{r}_i \tag{5}$$

With these simplifications, equation (2) becomes:

$$M\frac{d^2\vec{R}_{cm}}{dt^2} = \sum_{i=1}^{n}\vec{F}_i \tag{6}$$

which has the form of Newton's 2nd Law for a fictitious particle with mass M sitting at the center of mass of the system of particles and acted on by all of the *external* forces from the original system.

We can define the momentum of the center of mass as the total mass times the time derivative of the position of the center of mass:

$$\vec{P}_{cm} = M\frac{d\vec{R}_{cm}}{dt} \tag{7}$$

If there are no external forces acting, then the acceleration of the center of mass is zero and the momentum of the center of mass is constant in time (conserved).

$$M\frac{d^2\vec{R}_{cm}}{dt^2} = \frac{d\vec{P}_{cm}}{dt} = 0 \tag{8}$$

Notice that the entire discussion above applies even if all of the internal forces are zero $\vec{f}_{ij} = 0$, i.e. none of the particles have any way of knowing that the others are even present. Such particles are called non-interacting. The position of the center of mass of the system will still move according to equation (6).