Student handout

Expectation Values and Uncertainty

You have a system that consists of quantum particles with spin. On this system, you will perform a Stern-Gerlach experiment with an analyzer oriented in the z-direction.

Consider one of the different initial spin states described below:

A spin 1/2 particle described by:

1. $\left|+\right\rangle$
2. $\frac{i}{2}\left|+\right\rangle - \frac{\sqrt{3}}{2}\left|-\right\rangle$
3. $\left|+\right\rangle_x$

A spin 1 particle described by:

4. $\left|0\right\rangle$
5. $\left|-1\right\rangle_x$
6. $\frac{2}{3}\left|1\right\rangle + \frac{i}{3}\left|0\right\rangle - \frac{2}{3}\left|-1\right\rangle$

- List the possible values of spin you could measure and determine the probability associated with each value of the z-component of spin.
- Plot a histogram of the probabilities.
- Find the expectation value of the z-component of spin.
- Find the uncertainty of the z-component of spin.

1 Introduction

I like to break this activity into two parts:

(1) Calculating expectation values and relating them to the associated distributions of the probabilities of results, and

(2) Calculating the quantum uncertainty of the state and relating the uncertainty to distributions of the probabilities of results.

Therefore, I have my students do the first part of the activity before I introduce quantum uncertainty.

I introduce the activity by reminding students about two ways of calculating the expectation value. Given a quantum state $\left|\psi\right\rangle$, for a measurement of an observable represented by an operator $\hat{A}$ with eigenstates $\left|a_i\right\rangle$ and eigenvalues a_i:

$$\langle \hat{A} \rangle = \sum_i a_i \mathcal{P}(a_i)$$
$$= \langle \psi | \hat{A} | \psi \rangle$$

After the students calculate expectation values and we have a whole class discussion about 1 of the examples, then I do a lecture introducing the quantity of quantum uncertainty (relating it to the standard deviation of the distribution of probabilities by spin component value) and deriving the simplified equation:

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

2 Student Conversations

1. One could have each group report out, or the instructor could discuss a few key examples.

   For expectation value, I like to talk about Case 2: $\frac{i}{2}|+\rangle - \frac{\sqrt{3}}{2}|-\rangle$, where the probabilities of the two outcomes are not equal to show how the weighting plays out. Also, the expectation value is not a possible measurement value, and I like to talk about that. "Expectation" value is a misleading name for this quantity - it characterizes the distribution and is not necessarily a result of an individual measurement.

   I also like to discuss an example like Case 5: $|1\rangle_x$ where the distribution is symmetric around $0\hbar$.

2. I think it's important to encourage students to calculate expectation values both ways (with probabilities and as a bracket with matrix notation) while the teaching team is available to help them.

3. For quantum uncertainty, I like to talk about an example like Case 3: $|+\rangle_x$ where all the individual measurements are the same "distance" away from the expectation value as a sensemaking exercise to connect to a conceptual interpretation of physics.

   I also like to discuss an example like Case 5: $|-1\rangle_x$, where the fact that we're taking an rms average is apparent: half the measurements are $\hbar$ away from the expectation value and the other half are $0\hbar$ away, but the uncertainty is $\hbar/\sqrt{2}$.