1 Instructor's Guide

This activity is part of the Arms Sequence for Complex Numbers and Quantum States.

1.1 Prerequisite Ideas

Students should be familiar with spin-1/2 and spin-1 systems, particularly represently quantum states with Dirac notation, histograms, matrices, and arms.

1.2 Prompts:

1. **Set-Up**
   - Ask for a pair of students to demonstrate how to represent a quantum state with arms.
   - Each student represents one of the S_z basis states:
     person on the left $\rightarrow |+\rangle$
     person on the right $\rightarrow |-\rangle$
   - Each student uses their left arm to represent the complex probability amplitude for their basis ket.

2. Instructor writes a quantum state in Dirac notation on the board. "How would you represent this state using arms?"

   The instructor then asks the pair of volunteers (volunteers should write their answer on the board):
   - "What does this state look like as a histogram?"
   - "What does this state look like in matrix notation?"

3. Instructor asks."What would we have to do in order to represent a spin-1 state with arms?" Answer - add a student.
   - Instructor asks for another volunteer
   - Instructor writes a spin-1 state on the board and asks the student volunteers to represent it using: Arms, histogram, and matrix notations.

4. Instructor repeats exercise for a spin-3 system (7 basis kets)

5. Instructor asks the students to consider a new situation: "Now imagine that I have a particle and the position of the particle in an observable I'm interested in."
   - "How could we present a quantum state in the position basis with Arms?" Ask as many students as volunteer to join the Arms representation. Still not enough - we'd need an infinite number of people! And an infinite number of people in between each person.
   - What would the histogram look like? Answer: A function of x. This is the wavefunction.

- What would a quantum state written in a 1-D position basis look like? Answer: An infinitely long column with each entry corresponding a position eigenstate. The set of all the entries is the wavefunction.
- "What would the basis be like?"
  - (uncountably) infinite - even if the region is bounded. Between each person would be an infinite number of people.
  - spikey in space (like delta functions). Optional - you can foreshadow the uncertainty principle
  - in Dirac notation, label with the eigenvalue:
    $$\hat{X}\left|x\right\rangle = x\left|x\right\rangle$$
  - Complete - can write down a completion relation. The sum for a discrete basis, like spin, become an integral for a continuous basis:
    $$\sum_s \left|s\right\rangle\left\langle s\right| = 1$$
    $$\int \left|x\right\rangle\left\langle x\right| dx = 1$$
  - Talk about the dimensions of $\left|x\right\rangle \to 1/\sqrt{\text{length}}$

1.3 Wrap-up

Review points about:

- The position basis is continuous and (uncountably) infinite.
- Position basis kets are like spikey delta functions with dimensions of $1/\sqrt{\text{length}}$.

1.4 Extension

- Write a state in the position basis by applying a completeness relation:

$$
\begin{aligned}
|\psi\rangle &= \left(\int |x\rangle\langle x|\, dx\right)|psi\rangle \\
&= \int |x\rangle\langle x|\psi\rangle\, dx \\
&= \int \langle x|\psi\rangle\, |x\rangle\, dx \\
&= \int \psi(x)\, |x\rangle\, dx
\end{aligned}
$$

  Notice: $\langle x|\psi\rangle$ is the wavefunction. Because of the integral, we can't just take the norm squared of the wavefunction at a particular x to get the probability of the particle being at a particular location. The wavefunction doesn't have the right dimensions! Instead, the wavefunction is a "probability amplitude *density*". If you square the wavefunction, you get a probability density. You have to integrate the norm square of the wavefunction over a length to get a probability.

- In fact, asking the question "What is the probability of the particle being at a particular point?" is not a question we can answer with the formalism of quantum mechanics. We can only calculate the probability of a particle being in an infinitesimal region:

$$\mathcal{P}(\text{between } x_0 \text{ and } x_0 + dx) = |\langle x|\psi\rangle|^2 dx$$

  The multiplication by dx makes the quantity dimensionless.

  To calculate the probability of the particle being in a finite region:

$$\mathcal{P}(\text{between } x = a \text{ and } x = b) = \int_a^b |\langle x|\psi\rangle|^2 dx$$