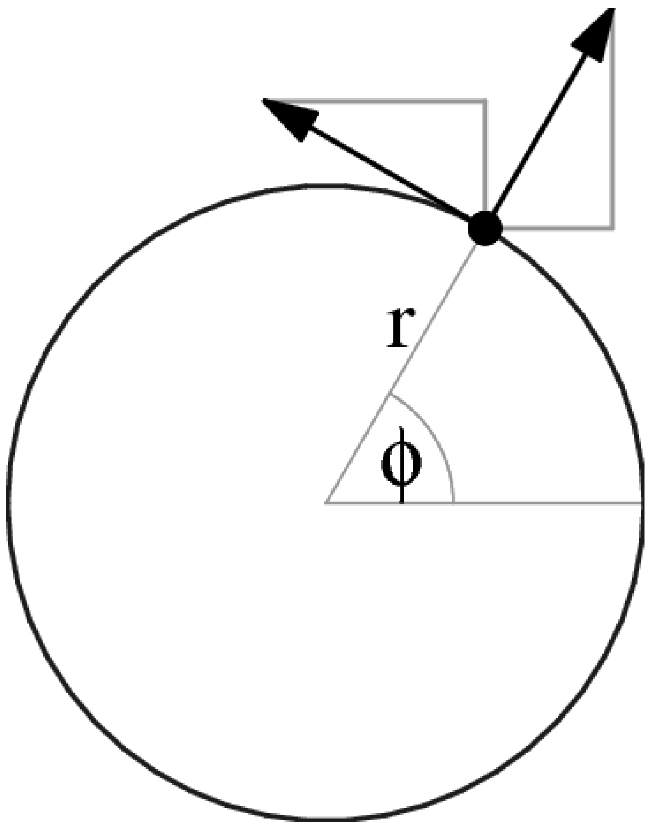
Student handout



In rectangular coordinates, the natural unit vectors are $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}\}$, which point in the direction of increasing x and y, respectively. Similarly, in polar coordinates the natural unit vectors are $\hat{\boldsymbol{r}}$, which points in the direction of increasing r, and $\hat{\boldsymbol{\phi}}$, which points in the direction of increasing ϕ.

The *unit tangent vector* to a parametric curve is the unit vector tangent to the curve which points in the direction of increasing parameter. The *principal unit normal vector* to a parametric curve is the unit vector perpendicular to the curve "in the direction of bending", which is the direction of the *derivative* of the unit tangent vector.

1. Consider the parametric curve $\vec{\boldsymbol{r}} = 3\cos\phi\,\hat{\boldsymbol{x}} + 3\sin\phi\,\hat{\boldsymbol{y}}$ with $\phi \in [0, 2\pi]$. Calculate the unit tangent vector $\hat{\boldsymbol{T}}$ and the principal unit normal vector $\hat{\boldsymbol{N}}$ for this curve in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.

2. Consider a circle of radius 3 centered at the origin. Determine the unit tangent vector $\hat{\boldsymbol{T}}$ and the principal unit normal vector $\hat{\boldsymbol{N}}$ for this curve in terms of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$.

3. Compare your answers.

1 Instructor's Guide

1.0.1 Main ideas

- Geometric introduction of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$.
- Geometric introduction of unit tangent and normal vectors.

1.0.2 Prerequisites

- The position vector $\vec{r}$.
- The derivative of the position vector is tangent to the curve.

1.0.3 Warmup

See the prerequisites. It is possible to briefly introduce these ideas immediately preceding this activity.

1.0.4 Props

- whiteboards and pens

1.0.5 Wrapup

Emphasize that $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ do not live at the origin! Encourage students to use the figure provided, which may help alleviate this confusion.

Point out to the students that $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are defined everywhere (except at the origin), whereas $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{N}}$ are properties of the curve. It is only on circles that these two notions coincide; $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are adapted to round problems, and circles are round! Symmetry is important.

Emphasize that $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}\}$ can be used as a basis (except at the origin). Point out to the students that their answer to the last problem gives them a formula expressing $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. When comparing these basis vectors, they should all be drawn with their tails at the same point.

1.1 Details

We have had success helping students master the idea of "direction of bending" by describing the curve as part of a pickle jar; the principal unit normal vector points at the pickles!

1.1.1 In the Classroom

The easiest way to find $\hat{\boldsymbol{N}}$ is to use the dot product to find vectors orthogonal to $\hat{\boldsymbol{T}}$, then normalize. Students must then use the "direction of bending" criterion to choose between the two possible orientations.

Finding $\hat{\boldsymbol{N}}$ in this way requires the student to give names to the its unknown components. This is a nontrivial skill; many students will have trouble with this.

It may be important to draw some examples. Despite that, students still feel wary of embracing $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$. People will feel more comfortable over the next few classes but emphasize the geometry: the circle is still the circle and the unit tangent remains the same regardless.

Some students are natural geometors and will realize what the desired vectors are. This is terrific. Certainly, it is worthwhile to explicitly demonstrate this, but the point is the geometry can often do the work for you. This will convince students of the value of smart coordinates.

1.1.2 Subsidiary ideas

- Dividing any vector by its length yields a unit vector.
- Using the dot product to find vectors perpendicular to a given vector.

1.1.3 Homework

- Some students will not be comfortable unless they work out the components of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ with respect to $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Let them.

1.1.4 Enrichment

- What units does a unit vector have? Do $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ have the same units?