Student handout

1. The diagonal of the rectangle on the left below shows (a blown-up picture of) an infinitesimal displacement from the point (x, y) to the nearby point
   $(x + dx, y + dy)$.
   - Label the rectangle with the lengths of the sides.
   - Express the sides of the rectangle indicated by arrows as vectors.
     Use the unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.
   - The diagonal of this rectangle is the vector differential $d\vec{r}$. Express $d\vec{r}$ in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.
   - Find the length $ds = |d\vec{r}|$ of the diagonal.

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2. The diagonal of the "rectangle" on the right above shows (a blown-up picture of) the *same* infinitesimal displacement, now expressed in polar coordinates, from the point (r, ϕ) to the nearby point $(r + dr, \phi + d\phi)$.
   - Label the rectangle with the lengths of the sides. *Careful!*
   - Express the sides of the rectangle indicated by arrows as vectors.
     Use the natural orthonormal basis defined by the picture, that is, let $\hat{r}$ be the unit vector which points in the direction of increasing $\vec{r}$, and let $\hat{\phi}$ be the unit vector which points in the direction of increasing ϕ. Do not attempt to express these vectors in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$! You do not need to worry about the fact that some sides of the rectangle aren't straight; the rectangle is so small that this error is negligible.
   - The diagonal of this rectangle is again the vector differential $d\vec{\boldsymbol{r}}$. Express $d\vec{\boldsymbol{r}}$ in terms of $\hat{r}$ and $\hat{\phi}$
   - Find the length $ds = |d\vec{r}|$ of the diagonal.

0.1 Essentials

0.1.1 Main ideas

- Introduces $d\vec{\boldsymbol{r}}$, the key to vector calculus, as a geometric object.

Don't skip this activity if you use nonrectangular basis vectors! [1]

[1] An alternative is to present this material in lecture, rather than as a group activity. In this case, we strongly recommend assigning the generalizations to cylindrical and spherical coordinates as homework.

0.1.2 Prerequisites

- Familiarity with $\hat{r}$ and $\hat{\phi}$. The Acceleration activity is a good introduction to those vectors.

0.1.3 Warmup

Draw a picture on the board showing $d\vec{r}$ as the infinitesimal change in the position vector $\vec{r}$ between two infinitesimally close points.

0.1.4 Props

- whiteboards and pens
- Big arrows, perhaps made of straws, which can represent an orthonormal basis, and which can be moved around a curve on the board.

0.1.5 Wrapup

- Emphasize that $d\vec{r}$ is the same geometric object regardless of how it is expressed.
- Discuss the geometry of ds as the magnitude of $d\vec{r}$, that is, $ds = |d\vec{r}|$.
- This is a good place to introduce the idea of "what sort of a beast is it". The vector differential $d\vec{r}$ is an infinitesimal differential having both direction and (infinitesimal) length. When writing an expression for $d\vec{r}$, students should make sure that each term has these same properties.

0.2 Details

0.2.1 In the Classroom

Most groups will miss the factor of r in the $\hat{\phi}$ component of $d\vec{r}$. Watch for this as you walk around the classroom. A good thing to point out is that $d\phi$ is not a length.

Some groups will then remember the formula for arclength and be able to figure out the rest on their own. Other groups will need to be reminded about the relationship between arclength and radius on a circle. A good way to do this is to ask them for the formula for the circumference of a circle, then half a circle, a quarter, etc. Make sure to give the angles in radians! Eventually, they get the point.

Some students may wonder whether the top of the (Cartesian) rectangle is $\pm dx\,\hat{x}$. This question is ill-posed, since the sign of dx itself depends on which way you're going; you can't change your mind in the middle of a problem. The safest way to resolve such problems is to anchor all vectors to the same point, as shown in the figures.

For the polar rectangle, many students will realize that that there are second-order differences between the two arcs, but few will realize that there are also second-order differences in the radial sides, due to changes in $\hat{r}$.

Plan to spend some extra time addressing the nature of $d\vec{r}$. Basis vectors, arc length, dot product and magnitude; there's a great deal to take in and it's easy to lose sight of the forest for the trees. People will benefit from a deeper understanding at this stage.

0.2.2 Subsidiary ideas

- This is a good place to emphasize the relationship between the dot product and the Pythagorean Theorem.

0.2.3 Homework

- Have students determine $d\vec{r}$ in 3 dimensions in rectangular, cylindrical and spherical coordinates. (Spherical coordinates are tricky; most students miss the factor of $\sin\theta$ in the $\vec{\phi}$ component.)
- Find $d\vec{r}$ along the diagonal of a square.

0.2.4 Enrichment

- Emphasize that $d\vec{r}$ is the concept which unifies most of vector calculus.
- It may be helpful to some students to be asked to orient the arrows (see Props) themselves at various points in the plane.