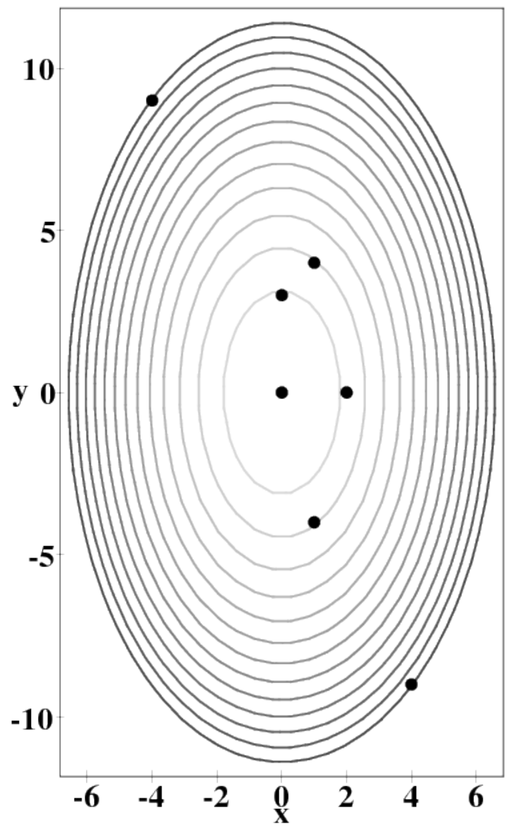
Student handout Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates (x, y) measured in miles. Your global positioning system says your present location is at one of the following points (pick one):



A: $(1, 4)$ **B:** $(4, -9)$ **C:** $(-4, 9)$ **D:** $(1, -4)$ **E:** $(2, 0)$ **F:** $(0, 3)$

Your guidebook tells you that the height h of the hill in feet above sea level is given by

$$h = a - bx^2 - cy^2$$

where $a = 5000\text{ft}$, $b = 30\,\frac{\text{ft}}{\text{mi}^2}$, and $c = 10\,\frac{\text{ft}}{\text{mi}^2}$.

- Starting at your present location, in what map direction (2-d unit vector) do you need to go in order to climb the hill as steeply as possible?

  Draw this vector on your topographic map.

> - How steep is the hill if you start at your present location and go in this compass direction?
>
>   *Draw a picture which shows the slope of the hill at your present location.*
>
> - In what direction in space (3-d vector) would you actually be moving if you started at your present location and walked in the map direction you found above?
>
>   *To simplify the computation, your answer does* ***not*** *need to be a unit vector.*

0.0.1 Main ideas

- Reinforce the geometric definition of the gradient.
- Differences between 2-d and 3-d representations of hills.
- Emphasize that the gradient lives in the domain, not on the graph.

0.0.2 Prerequisites

- The ability to compute the gradient of a function.
- The geometric interpretation of the gradient, that is, that the direction of the gradient of a function gives the direction of greatest increase of the function, while the magnitude gives the rate of increase at that point in that direction.

0.0.3 Warmup

- It takes the students surprisingly long to draw the topo map themselves. If you value this skill, you can speed thing up by having them sketch it beforehand. Otherwise, you can supply the contour graph.

0.0.4 Props

- whiteboards and pens
- any topographic map
- hill transparency
- blank transparencies and pens

0.1 Details

It's tempting to use a hill as a nice geometric example of a function of two variables. However, doing so opens a can of worms. In examples like this, when the function has dimensions of length, students are confused as to whether the gradient is 2-dimensional or 3-dimensional. In most applications, involving physical quantities such as temperature, this confusion does not arise. If you want to use hills as an important example, then it's best to confront this confusions head-on; this lab is a good way to do so, although this requires fairly sophisticated geometric reasoning. If you choose to restrict to other applications, you may prefer to skip this lab.

0.1.1 In the Classroom

- You may wish to have students work on the points on the axes (**E** or **F**) first, then try another point.
- If you are not using an overhead projector, draw the topo map on the board while students are working on this activity so it's available for the wrapup.
- If you are using an overhead, put a blank transparency over the hill master and have students draw on that. If your classroom has whiteboards rather than blackboards, you can also project the transparency directly onto the board, then have students draw on the board.
- In the last question, groups may get the scale wrong. If the horizontal part of their answer is a unit vector, then they should use the magnitude of the gradient as the z-component. But if the horizontal part of their answer is the gradient, then they must scale the z-component appropriately.

0.1.2 Wrapup

- Have each group draw their gradient vector on the board on a single topo map. This is a good place to introduce the term "vector field".
- **Where does the gradient live?** Students will not realize that, because the height is a function of two variables in this problem, the gradient of the height function is a 2-D vector that lives in the topo map.
- **Compass direction versus slope:** The gradient tells the students the direction of steepest ascent, and it also contains information about how quickly the height function is changing.

  $$df = \vec{\nabla} f \cdot d\vec{r}$$

  Therefore, the magnitude of the gradient is the slope. To find the 3-D vector direction of travel, students need to find a unit vector in the direction of gradient as well as the change in height. Most students will forget that they need a normalized vector in the x-y plane to give the 3-D vector pointing along the steepest direction at their point.
- See Acting Out the Gradient. Ask students to stand up and imagine that they are standing at a point on the hill. Have them point in the direction of the the gradient. Many will point "uphill", thinking that the gradient is 3-dimensional even though their computed answer does not contain $\hat{z}$. Discuss this! The key understanding is that the gradient is always perpendicular to the level curves(for two dimensions) that they lie on. This is regardless of where the global maximum is of the function: if the level curves are not circular, then not everyone will be pointing towards the same point.

0.1.3 Homework

Consider a valley whose height h in meters is given by $h = \frac{x^2}{10} + \frac{y^2}{10}$, with x and y (and 10!) in meters. Suppose you are hiking through this valley on a trail given by $x = 3t$, $y = 2t^2$, with t in seconds. How fast are you climbing *per meter* along the trail when $t = 1$? How fast are you climbing *per second* when $t = 1$.

0.1.4 Enrichment

- What is the length of the vector in the last problem. What does it mean? Do the units need to be the same in each term?

- Discuss which way you should go to get to the top of the hill the fastest. What does this mean? The shortest path (geodesic)? The one with the largest *average* steepness? The *smallest* average steepness?

  As one student put it, the answer depends not only on the shape of the hill, but also on the shape of the hiker!

- Students may find it easier to visualize what it means to go *down* as steeply as possible. If you empty your canteen on the ground, which way would the water go? If you were skiing, which way would you go?

- Why is it that on a real hill, turning exactly 90° from the steepest direction always keeps you at constant height?

- Regard the hill as a level surface of the function of three variables $z - h(x, y)$. What is the gradient of this function? What does it mean geometrically? How is it related to the gradient of h and to the 3-dimensional vector found in the last problem?

  (This is a good problem for honors students.)

- Point out that if the drawing is not to scale — as produced for instance by the default settings for both Maple and Mathematica — then the gradient will not appear to be perpendicular to the level curves!

- Directional derivatives are rates of change with respect to "distance traveled". It is important to realize that the steepness of the hill is the directional derivative with respect to distance traveled *in the topo map* (the "run"), not physical distance traveled on the hill.