Student handout You are in a valley whose height is given by $h = ax^2 + ay^2$ where $a = \frac{1}{10}\frac{\text{ft}}{\text{mi}^2}$.

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Your location corresponds to $x = y = 1$ mi. Your goal is to reach the road located at $y = 0$.

- Choose *one* of the following paths, and sketch it on your map.

  I: $x^2 + y^2 = 2$ **II:** $y = x$ **III:** $y = x^2$ **IV:** $(y-1) = 3(x-1)$ **V:** $x = 1$

- Determine $\vec{\nabla} h$ at your location.
- Calculate $\oint_C \vec{\nabla} h \cdot d\vec{r}$ along your path.
- Compute $\int_C dh$ along your path.
- Compare your answers to these two integrals. What do your answers represent?
  Is there an easier way to get the same answer?

0.1 Essentials

0.1.1 Main ideas

- Reinforces both the Master Formula and differentials.
- Sets the stage for path-independence.

0.1.2 Prerequisites

- Some familiarity with differentials.
- Familiarity with the gradient.

0.1.3 Warmup

A brief derivation of the master formula from the expression for the differential of a function of two variables.

0.1.4 Props

- whiteboards and pens
- valley transparency
- blank transparencies and pens

0.1.5 Wrapup

- Call someone from each group to the board to draw both their path and $d\vec{\boldsymbol{r}}$ on the topo map and show how they found $d\vec{\boldsymbol{r}}$. Discuss the different methods used by different groups. The idea here is that **on a curve** dy is related to dx. Students are being asked to find this relationship, and plug it into the general expression for $d\vec{\boldsymbol{r}}$. *"Use what you know! Any (algebraically correct) method will work."*

- Emphasize that ∇h is a property of the hill, while $d\vec{\boldsymbol{r}}$ is a property of the curve. The point of the master formula is that it naturally separates the information in dh into these quite different geometric ideas.

- Have the class discuss why the answer to the second integral is in fact easy to find without integration.

0.2 Details

0.2.1 In the Classroom

- This lab is on the long side; don't plan to do *anything* else in a 50-minute period. The wrapup alone easily requires 20 minutes to do properly; you may wish to do part of it in a subsequent class period.
- Some students may not realize that $(1, 1)$ is on the given circle!
- Ask the students if their level curves are equally spaced.

  (They shouldn't be.)
- Initially assign each group one of the curves; groups which finish quickly can try other curves. The first curve, the circle, is qualitatively different from the others, and more difficult. Furthermore, the instructions do not uniquely determine the curve in this case — although the final answer is unaffected. You may wish to assign this curve to a strong group, or not let any group try the circle until they have first done one of the other curves.
- Some students substitute the given point into the height function before computing the gradient! Perhaps asking for a sketch of ∇h at several points rather than just one would discourage this.
- Ensure that students reduce to one variable before integrating.
- Emphasize that one can plug in the relationship between x and y either before or after computing the differential of h. Which choice is easiest depends on the circumstances; both will work.
- On path IV, it is indeed necessary to solve for y at some point even if using differentials; some students may prefer to do this first, effectively parameterizing the curve with respect to x.
- In the next-to-last question, groups may need to be reminded that they need to plug in information about their curve in order to find dh. They should use the expression for the differential of h as a function of either one or two variables, rather than the master formula (which should not be used until the last question).
- Some students will realize that the integrals must be the same because of the master formula before ever trying to compute the second integral. Such students should be praised — but still encouraged to compute the second integral without using the master formula.
- On the circle, some students go from $x^2 + y^2 = a^2$ directly to $d\vec{\boldsymbol{r}} = 2x\,dx\,\hat{\boldsymbol{x}} + 2y\,dy\,\hat{\boldsymbol{y}}$ One way to push students away from this mistake is to emphasize that one *always* has $d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}}$ (or a similar expression in other coordinate systems). We literally stomp our feet when insisting that students start problems involving $d\vec{\boldsymbol{r}}$ by writing down one of these expressions! A discussion of this point works well as part of the wrapup.
- Emphasize that $\oint \nabla h \cdot d\vec{\boldsymbol{r}}$ is a definite integral, and that $\oint 0\,dx = 0$ (not 1).
- See the discussion of using transparencies for The Hill Group Activity.

0.2.2 Subsidiary ideas

- The gradient is perpendicular to level curves.
- Emphasize that $df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$ is a coordinate-dependent expression for df, whereas writing $df = \nabla f \cdot d\vec{\boldsymbol{r}}$ is coordinate independent.

0.2.3 Homework

1. Consider the valley in this group activity, whose height h in meters is given by $h = \frac{x^2}{10} + \frac{y^2}{10}$, with x and y (and 10!) in meters. Suppose you are hiking through this valley on a trail given by

$$x = 3t \qquad y = 2t^2$$

with t in seconds (and where "3" and "2" have appropriate units).

   a) Starting from the master formula, determine how fast you are climbing (rate of change of h) *per meter* along the trail when $t = 1$. *You may find it helpful to recall that* $ds = |d\vec{\boldsymbol{r}}|$.

   b) Starting from the master formula, determine how fast you are climbing *per second* when $t = 1$.

0.2.4 Essay questions

- During this activity, you drew a gradient vector on a topographic map. Can you draw this vector to scale? Explain.
- What properties of your path are needed to compute the integrals in this activity? To determine the answer?

0.2.5 Enrichment

- Discuss the relationship between the master formula, the gradient, topographic maps, and path-independence.
- Discuss the fundamental theorem for gradients, namely that the line integral of a gradient is just an obvious antiderivative. Relate this to the geometry, especially the existence of a topo map.
- Many students will integrate the two pieces of $dh = 2x\,dx + 2y\,dy$ separately, without worrying about the path. What path is implicitly being used? Does it matter?
- We strongly discourage students from inserting artificial signs into expressions such as $d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}}$. This forces $dy < 0$, and in some cases also $dx < 0$, so that one must integrate from 1 to 0. By all means discuss the alternative convention with students, which requires dx and dy to always be positive, and then forces one to insert (and keep track of) appropriate signs by hand.
- Following this lab is a good time to introduce or review the proof, using the master formula, that the gradient is perpendicular to level curves and that it points in the direction of maximal increase.
- A great followup to this activity is a discussion of what questions you can answer using the master formula.

- It is immediately obvious in polar coordinates that these integrals do not depend on ϕ, and hence are independent of path.