Student handout Consider the vector field given by *(μ_0 and I are constants)*: $\vec{\boldsymbol{B}} = \frac{\mu_0 I}{2\pi}\left(\frac{-y\,\hat{\boldsymbol{x}}+x\,\hat{\boldsymbol{y}}}{x^2+y^2}\right) = \frac{\mu_0 I}{2\pi}\,\frac{\hat{\boldsymbol{\phi}}}{s}$

$\vec{\boldsymbol{B}}$ is the magnetic field around a wire along the z-axis carrying a constant current I in the z-direction.

Ready:

- Determine $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ on any radial line of the form $y = mx$, where m is a constant.
- Determine $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ on any circle of the form $x^2 + y^2 = a^2$, where a is a constant.
  You may wish to express the equations for these curves in polar coordinates.

Go: For each of the following curves C_i, evaluate the line integral $\int_{C_i} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$.

- C_1, the *top* half of the circle $s = 5$, traversed in a *counterclockwise* direction.
- C_2, the *top* half of the circle $s = 2$, traversed in a *counterclockwise* direction.
- C_3, the *top* half of the circle $s = 2$, traversed in a *clockwise* direction.
- C_4, the *bottom* half of the circle $s = 2$, traversed in a *clockwise* direction.
- C_5, the radial line from $(2, 0)$ to $(5, 0)$.
- C_6, the radial line from $(-5, 0)$ to $(-2, 0)$.

FOOD FOR THOUGHT

- Construct ***closed*** curves C_7 and C_8 such that this integral $\int_{C_i} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ is nonzero over C_7 and zero over C_8.
  *It is enough to draw your curves; you do **not** need to parameterize them.*
- Ampère's Law says that, for any closed curve C, this integral is (μ_0 times) the current flowing ***through*** C (in the z direction). Can you use this fact to explain your results to part (a)?
- Is $\vec{\boldsymbol{B}}$ conservative?

0.0.1 Main ideas

- Calculating (vector) line integrals.
- Use what you know!

0.0.2 Prerequisites

- Familiarity with $d\vec{\boldsymbol{r}}$.

- Familiarity with "Use what you know" strategy.

0.0.3 Warmup

This activity should be preceded by a short lecture on (vector) line integrals, which emphasizes that $\int_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ represents chopping up the curve into small pieces. Integrals are sums; in this case, one is adding up the component of $\vec{\boldsymbol{B}}$ parallel to the curve times the length of each piece.

0.0.4 Props

- whiteboards and pens

0.0.5 Wrapup

Emphasize that students must express everything in terms of a single variable prior to integration.

Point out that in polar coordinates (and basis vectors)

$$\vec{\boldsymbol{B}} = \frac{\mu_0 I}{2\pi} \frac{\hat{\boldsymbol{\phi}}}{s}$$

so that using $d\vec{\boldsymbol{r}} = ds\,\hat{\boldsymbol{s}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$ quickly yields $\vec{\boldsymbol{B}} \cdot dd\vec{\boldsymbol{r}}$ along a circular arc ($\frac{\mu_0 I}{2\pi}\,d\phi$) or a radial line (0), respectively.

0.1 Details

0.1.1 In the Classroom

- Sketching the vector field takes some students a long time. If time is short, have them do this before class, or consider using MATLAB or similar technology to plot the field. Still, it's important to plot a few vectors by hand.
- Students who have not had physics don't know which way the current goes; they may need to be told about the right-hand rule.
- Some students may confuse the wire with the paths of integration.
- Students working in rectangular coordinates often get lost in the algebra of Question 2b. Make sure that nobody gets stuck here.
- Students who calculate $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \frac{dy}{x}$ on a circle need to be reminded that at the end of the day a line integral must be expressed in terms of a single variable.
- Some students will be surprised when they calculate $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = 0$ for radial lines. They should be encouraged to think about the directions of $\vec{\boldsymbol{B}}$ and $d\vec{\boldsymbol{r}}$.
- Most students will either write everything in terms of x or y or switch to polar coordinates. We discuss each of these in turn.
  - This problem cries out for polar coordinates. Along a circular arc, $s = a$ yields $x = a\cos\phi$, $y = a\sin\phi$, so that $d\vec{\boldsymbol{r}} = -a\sin\phi\, d\phi\, \hat{\boldsymbol{x}} + a\cos\phi\, d\phi\, \hat{\boldsymbol{y}}$, from which one gets $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \frac{\mu_0 I}{2\pi}\, d\phi$.
  - Students who fail to switch to polar coordinates can take the differential of both sides of the equation $x^2 + y^2 = a^2$, yielding $x\,dx + y\,dy = 0$, which can be solved for dx (or dy) and inserted into the fundamental formula $d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}}$. Taking the dot product then yields, $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \frac{\mu_0 I}{2\pi}\frac{dy}{x}$. Students may get stuck here, not realizing that they need to write x in terms of y. The resulting integral cries out for a trig substitution — which is really just switching to polar coordinates.

  In either case, sketching $\vec{\boldsymbol{B}}$ should convince students that $\vec{\boldsymbol{B}}$ is tangent to the circular arcs, hence orthogonal to radial lines. Thus, along such lines, $\vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = 0$; no calculation is necessary. (This calculation is straightforward even in rectangular coordinates.)
- Watch out for folks who go from $s^2 = x^2 + y^2$ to $d\vec{\boldsymbol{r}} = 2x\,dx\,\hat{\boldsymbol{x}} + 2y\,dy\,\hat{\boldsymbol{y}}$.
- Working in rectangular coordinates leads to an integral of the form $\int -\frac{dx}{y}$, with $y = \sqrt{s^2 - x^2}$. Maple integrates this to $-\tan^{-1}\left(\frac{x}{y}\right)$, which many students will not recognize as the polar angle ϕ. If $s = 1$, Maple instead integrates this to $-\sin^{-1} x$; same problem. One calculator (the TI-89?) appears to use arcsin in both cases.

0.1.2 Subsidiary ideas

- Independence of path.

0.1.3 Homework

- Any vector line integral for which the path is given geometrically, that is, without an explicit parameterization.

0.1.4 Essay questions

- Discuss when $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ around a closed curve will or will not be zero.

0.1.5 Enrichment

- This activity leads naturally into a discussion of path independence.
- Point out that $\vec{\boldsymbol{B}} \sim \nabla\phi$ everywhere (except the origin), but that $\vec{\boldsymbol{B}}$ is only conservative on domains where ϕ is single-valued.
- Discuss *winding number*, perhaps pointing out that $\vec{\boldsymbol{B}} \cdot d\hat{\boldsymbol{r}}$ is proportional to $d\phi$ along *any* curve.
- Discuss *Ampère's Law*, which says that $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}}$ is (μ_0 times) the current flowing *through* C (in the z direction).