Student handout

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1. Consider the rectangle in the first quadrant of the xy-plane as in the figure with thick black lines.
   - Label the bottom horizontal edge of the rectangle $y = c$.
   - Label the sides of the rectangle Δx and Δy.
   - What is the area of the rectangle?
   - There are also 2 rectangles whose base is the x-axis, the larger of which contains both the smaller and the original rectangle. Express the area of the original rectangle as the difference between the areas of these 2 rectangles.
2. On the grid below, draw any simple, closed, piecewise smooth curve C, all of whose segments C_i are parallel either to the x-axis or to the y-axis. Your curve should **not** be a rectangle. Pick an origin and label it, and assume that each square is a unit square.

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   - Compute the area of the region D inside C by counting the number of squares inside C.
   - Evaluate the line integral $\oint_C y\,\hat{\boldsymbol{x}} \cdot d\vec{\boldsymbol{r}}$ by noticing that along each segment either x or y is constant, so that the integral is equal to $\sum_{C_i} y\,\Delta x$.
     Can you relate this to Problem 1?
   - Are your answers to the preceding two calculations the same?
   - Would any of your answers change if you replaced $y\,\hat{\boldsymbol{x}}$ by $x\,\hat{\boldsymbol{y}}$ in part (b)?

0.0.1 Main ideas

- Understanding different ways of expressing area using integration.
- Concrete example of Area Corollary to Green's/Stokes' Theorem.

We originally used this activity after covering Green's Theorem; we now skip Green's Theorem and do this activity shortly before Stokes' Theorem.

0.0.2 Prerequisites

- Familiarity with line integrals.
- *Green's Theorem is not a prerequisite!*

0.0.3 Warmup

- The first problem is a good warmup.

0.0.4 Props

- whiteboards and pens
- a planimeter if available

0.0.5 Wrapup

- Emphasize the magic – finding area by walking around the boundary!
- Point out that this works for any closed curve, not just the rectangular regions considered here.
- Demonstrate or describe a planimeter, used for instance to measure the area of a region on a map by tracing the boundary.

0.1 Details

0.1.1 In the Classroom

- Make sure students use a consistent orientation on their path.
- Make sure students explicitly include all segments of their path, including those which obviously yield zero.
- Students in a given group should all use the same curve.
- Students should be discouraged from drawing a curve whose longest side is along a coordinate axis.
- Students may need to be reminded that $\oint$ implies the counterclockwise orientation. But it doesn't matter what orientation students use so long as they are consistent!
- A geometric argument that the orientation should be reversed when interchanging x and y is to rotate the xy-plane about the line $y = x$. (This explains the minus sign in Green's Theorem.)
- Students may not have seen line integrals of this form (see below).
- Students do very well on this lab, particularly after working in groups for several weeks. Resist the urge to intervene.
- Make sure everyone sees the reason $y\,\hat{\boldsymbol{x}} \cdot d\vec{\boldsymbol{r}}$ is zero on vertical pieces.
- The issue of the negative will come up. Suggest students make a quick sketch of the vector field.
- It is well worthwhile to do an example with a circle together as a class. The line integral should pose no trouble for them and the area of a circle is something they believe.
- Emphasize the connection between the boundary and the interior. This is a concrete display of this relationship.

0.1.2 Subsidiary ideas

- Orientation of closed paths.
- Line integrals of the form $\int P\,dx + Q\,dy$.

  We do not discuss such integrals in class! Integrals of this form almost always arise in applications as $\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$.

0.1.3 Homework

Determine the area of a triangle or an ellipse by integrating along the boundary.

0.1.4 Essay questions

Describe times in your life when you needed to know area (or imagine such a time). Maybe buying carpet or painting a room. What is the first step in computing area? How does this lab truly differ, if at all?

0.1.5 Enrichment

- Write down Green's Theorem.
- Go to 3 dimensions — bend the curve out of the plane and stretch the region like a butterfly net or rubber sheet. This is the setting for Stokes' Theorem!