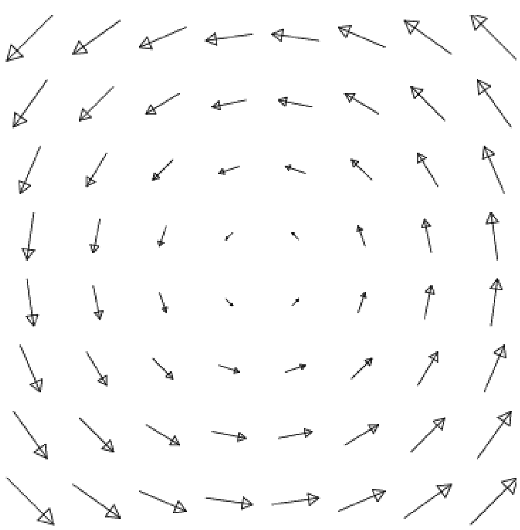
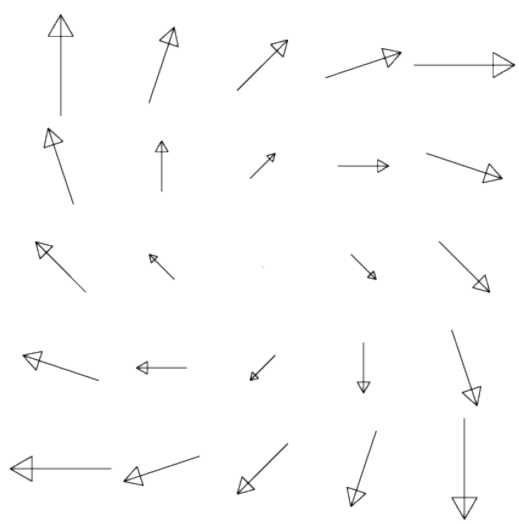
Student handout Choose a vector field $\vec{\boldsymbol{F}}$ from the first column below. Choose a small loop C (that is, a simple, closed, positively-oriented curve) which does **not** go around the origin.

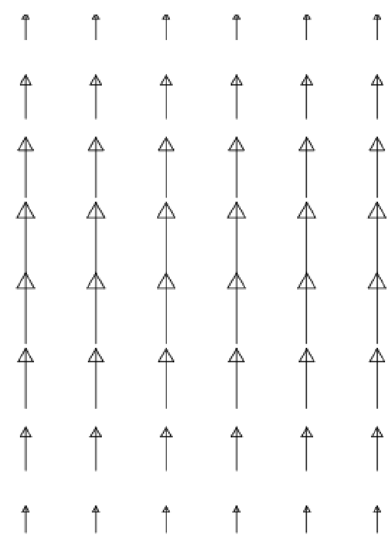
- Is $\oint \vec{\boldsymbol{F}} \cdot d\hat{\boldsymbol{r}}$ positive, negative, or zero?
- Will a paddlewheel spin if placed inside your loop, and, if so, which way?
- Do you think $\nabla \times \vec{\boldsymbol{F}}$ is zero or nonzero inside your loop?
  Explain.
- Compute $\nabla \times \vec{\boldsymbol{F}}$. Did you guess right? *Explain.*
- Is $\oint \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, ds$ positive, negative, or zero? ($\hat{\boldsymbol{n}}$ is the *outward* pointing normal vector to C.)
- Is the net flow outwards across your loop positive, negative, or zero?
- Do you think $\nabla \cdot \vec{\boldsymbol{F}}$ is zero or nonzero inside your loop? *Explain.*
- Compute $\nabla \cdot \vec{\boldsymbol{F}}$. Did you guess right? *Explain.*
- Repeat the above steps for vector fields $\vec{\boldsymbol{G}}$ and $\vec{\boldsymbol{H}}$ chosen from the second and third columns.



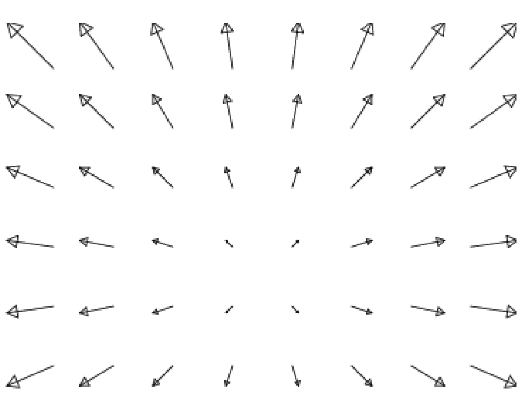
$-y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}}$

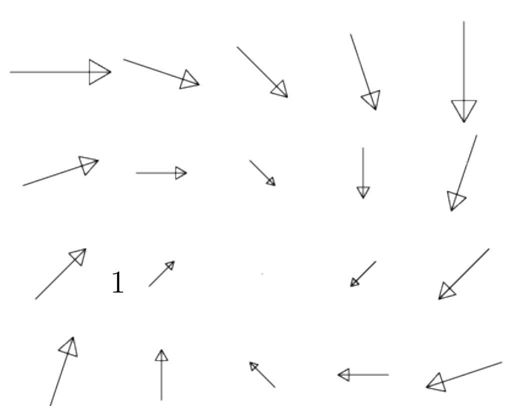


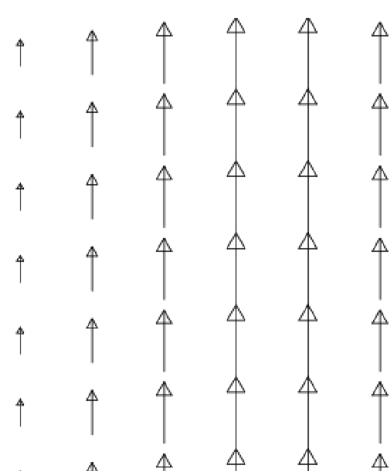
$(x+y)\,\hat{\boldsymbol{x}} + (y-x)\,\hat{\boldsymbol{y}}$



$e^{-y^2}\,\hat{\boldsymbol{y}}$







0.0.1 Main ideas

- Visualization of divergence and curl.

0.0.2 Prerequisites

- Definition of divergence and curl.
- Geometry of divergence and curl, either through a geometric definition or through Stokes' Theorem and the Divergence Theorem.

0.0.3 Warmup

- Students may need to be reminded what *circulation* is.
- Students may not have seen *flux* in 2 dimensions.
- Students may only have seen $\hat{\boldsymbol{n}}$ for surfaces, not curves. Some students will set $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$! Emphasize that $\hat{\boldsymbol{n}}$ is horizontal (and that $ds \neq \boldsymbol{d\vec{S}}$).

0.0.4 Props

- whiteboards and pens
- formula sheet for div and curl in spherical and cylindrical coordinates (Each group may need its own copy.)
- divergence and curl transparency
- blank transparencies and pens

0.0.5 Wrapup

- Discuss the effect of choosing loops of different shapes, especially those adapted to the given vector field.
- Talk about the geometry of sinks and sources (for divergence) and paddlewheels (for curl).

0.1 Details

0.1.1 In the Classroom

- While students are working on this activity, draw the vector fields on the board to use during the wrapup. Alternatively, bring an overhead transparency showing the vector fields (and blank transparencies for students to write on).
- Students like this lab; it should flow smoothly and quickly.

- Students may need to be reminded what $\oint$ means, and that the positive orientation in the plane is counterclockwise.
- Yes, two pairs of questions are really the same.
- Make sure the paths do *not* go around the origin.
- Encourage each group to work on at least two vector fields, which are in different rows and columns. Include one vector field from the third column if time permits.
- Encourage each group to consider, for a single vector field, moving their loop to another location. This is especially effective (and in fact essential) for the two vector fields in the third column.
- See the discussion of using transparencies for Group Activity The Hill.
- Students may eventually realize that the vector fields in the middle column are linear combinations of the vector fields in the first column, which are in turn "pure curl" and "pure divergence", respectively.

0.1.2 Subsidiary ideas

- Divergence and curl are not just about the behavior near the origin. Derivatives are about *change* — the *difference* between nearby vectors.

0.1.3 Homework

(MHG refers to McCallum, Hughes Hallett, Gleason, et al.

- MHG 19.1:20
- MHG 20.2:16
- MHG 20.3:10,12,20
- MHG 20.4:22

0.1.4 Essay questions

- Which operation, curl or divergence is easier to understand?
- Which is more useful?
- Do you prefer to gauge curl from a plot or from a calculation? What about divergence?

0.1.5 Enrichment

- Emphasize the importance of divergence and curl in applications.
- Ask students how to determine which vector fields are conservative! (A single closed path with nonzero circulation suffices to show that a vector field is *not* conservative. The best geometric way we know to show that a vector field *is* conservative is to try to draw the level curves for which the given vector field would be the gradient.)
- Discuss the fact that $\frac{\hat{\boldsymbol{r}}}{r}$ and $\frac{\hat{\boldsymbol{\phi}}}{r}$ are *both* curl-free and divergence-free; this is counterintuitive, but crucial for electromagnetism. (These are, respectively, the electric/magnetic field of a charged/current-carrying wire along the z-axis.)
- Discuss the behavior of $\frac{\hat{\boldsymbol{r}}}{r^n}$ and $\frac{\hat{\boldsymbol{\phi}}}{r^n}$, emphasizing that *both* the divergence and curl vanish when $n = 1$.
- Relate these examples to the magnetic field of a wire ($\vec{\boldsymbol{B}} = \frac{\hat{\boldsymbol{\phi}}}{r}$) and the electric field of a point charge ($\vec{\boldsymbol{E}} = \frac{\hat{\boldsymbol{r}}}{r^2}$; this is the *spherical* r).
- Show students how to compute divergence and curl of these vector fields in cylindrical coordinates.
- Trying to estimate divergence and curl from a single plot of a vector field confronts students with the need to zoom in. Technology can be useful here.
- Point students to our paper on *Electromagnetic Conic Sections*, which appeared in Am. J. Phys. **70**, 1129–1135 (2002), and which is also available on the Bridge Project website.
- Most physical applications of the divergence are 3-dimensional, rather than 2-dimensional. Each vector field in this activity could be regarded as a horizontal 3-dimensional vector field by assuming that there is no z-dependence, in which case the flux can be computed through a 3-dimensional box whose cross-section is the loop, and whose horizontal top and bottom do not contribute.