Student handout An ice cream cone is to be dipped in chocolate. The cone can be described by the equation $z^2 = 9\,(x^2 + y^2)$, with $0 \le z \le 9$ and x, y, and z in centimeters. The dipping process is such that the resulting (surface) density of chocolate on the cone is given by $\sigma = 1 - \frac{z}{9}$ in grams per square centimeter. Find the total amount of chocolate on the cone.

(There is no ice cream on the cone!)

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0.1 Essentials

0.1.1 Main ideas

- Calculating (scalar) surface integrals.
- Use what you know!

0.1.2 Prerequisites

- Familiarity with (vector) surface elements in the form $d\vec{\boldsymbol{A}} = d\vec{\boldsymbol{r}}_{\mathbf{1}} \times d\vec{\boldsymbol{r}}_{\mathbf{2}}$.

0.1.3 Warmup

It is *not* necessary to explicitly introduce scalar surface integrals, before this lab; figuring out that the (scalar) surface element must be $|d\vec{\boldsymbol{r}}_{\mathbf{1}} \times d\vec{\boldsymbol{r}}_{\mathbf{2}}|$ can be made part of the activity (if time permits). (We have done it both ways successfully.)

0.1.4 Props

- whiteboards and pens
- any chocolate covered candy

0.1.5 Wrapup

Emphasize that the formula for the surface area of the cone is *not* helpful.

Emphasize that one must choose between r and z prior to integration. When doing a double integral, everything must be expressed in terms of precisely two variables.

This lab can also be done using

$$dA = |d\vec{\boldsymbol{A}}| = |d\vec{\boldsymbol{r}}_{\mathbf{1}}|\,|d\vec{\boldsymbol{r}}_{\mathbf{2}}|\sin\theta = (r\,d\phi)\left(\sqrt{dr^2 + dz^2}\right)$$

0.2 Details

0.2.1 In the Classroom

- Watch out for students who write $\vec{\boldsymbol{r}}_1 \times \vec{\boldsymbol{r}}_2$ instead of $d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2$; this can be due to misreading the (traditional) text.
- This problem cries out for cylindrical coordinates, in which the equation of the cone is simply $z = 3s$. This makes it straightforward to use either z or s (and ϕ) as the integration variables. But rectangular coordinates also work fine.
- Some students will try to use geometry to determine $d\vec{\boldsymbol{A}}$. Such students will often reduce the problem to a single integration by chopping the cone into bands, whose area is $2\pi s$ times the width of the band. But many students will have trouble seeing that this width is not dz! One argument which often helps is to compute $ds = |d\vec{\boldsymbol{r}}|$ along a line with ϕ=constant.
- Ask students who do a single integral what they would do if the density σ depended on ϕ.
- Some students may worry about the lack of a normal vector at the tip of the cone. It's fairly clear that this isn't a problem when determining how much chocolate is on the cone, but it is less obvious that this also causes no difficulty for (some other problem involving) flux. The key idea is that the affected region is small — a set of measure zero.
- The Cone follows The Pretzel very well. Not much explanation is needed about mass and the magnitude of the vector differential. It is important to emphasize the analogous ideas.
- Students are inclined to use curvilinear basis vectors here if they've been introduced. Some groups notice that ϕ is not present in the defining equation and deduce that $d\phi = 0$. Question people about the effect of constant ϕ on the surface.
- Afterward, ask questions about where $d\vec{\boldsymbol{A}}$ is pointing and how to make this surface closed. One can even have a vector field such as $s\hat{\boldsymbol{s}}$ in mind for a practice flux integral.
- Some people asked if there is chocolate on the inside of the cone as well. Wishful thinking! The density function shows that the chocolate tapers off toward the top suggesting that the cone was dipped tip first.

0.2.2 Subsidiary ideas

- $d\vec{\boldsymbol{A}} = |d\vec{\boldsymbol{A}}| = |d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2|$