Student handout

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A fishing net S is in the shape of a triangular trough, as shown in the picture. The triangular sides are at $x = 0$ and $x = 5$, the rectangular sides are at 45° to the vertical, and the bottom is at $z = 0$; all lengths are measured in cm. There is no netting across the top, which is at $z = 1$. Water is draining out of the net; the motion of the water is described by the vector field $\vec{\boldsymbol{F}} = \rho\left(ae^{\kappa z^2}\hat{\boldsymbol{y}} - b\,\hat{\boldsymbol{z}}\right)$ where $a = 3\,\frac{\text{cm}}{\text{s}}$, $b = 5\,\frac{\text{cm}}{\text{s}}$, $\kappa = 2\,\text{cm}^{-2}$, and ρ is the (constant) density of the water in $\frac{\text{g}}{\text{cm}^3}$. The goal of this problem is to find the best way to evaluate the flux

$$\int\!\!\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}}$$

of water *down* through S.

- Set up the above surface integral, **but do not evaluate it**
  Your answer should be ready to integrate; among other things, all substitutions should be made, and you should determine the correct limits.
- **Use the Divergence Theorem** to find another way to do the problem.
  This time, complete the computation.

0.0.1 Main ideas

- Practice doing surface integrals
- The Divergence Theorem

0.0.2 Prerequisites

- Ability to do flux integrals
- Definition of divergence
- Statement of Divergence Theorem *This lab can be used prior to covering the Divergence Theorem in class with either a minimal introduction or a restatement of the last question based on the assumption that the given vector field doesn't "lose" anything going through the net.*

0.0.3 Warmup

- Perhaps a reminder about what the Divergence Theorem is.

0.0.4 Props

- whiteboards and pens
- a model of the fishing net, made from any children's building set

0.0.5 Wrapup

- Reiterate that the Divergence Theorem only applies to closed surfaces.
- Emphasize that the Divergence Theorem is one of several astonishing theorems relating what happens inside to what happens outside.
- Have several students show how they computed $d\vec{S}$, since most likely different choices were made for $d\vec{r}_i$ and hence the limits.

0.1 Details

0.1.1 In the Classroom

- By now the groups should be working well. Sit back and watch!
- The main thing to watch out for is whether students choose the correct signs, both for the normal vectors and the limits of integration. Reiterate that one should *always* write $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$; there should *never* be minus signs in this equation. The signs will come out right provided one integrates in the direction of the vectors chosen.
- Most students will realize quickly that there is no flux through the triangular sides. (encourage supporting this with geometry and a calculation).
- Some students will try to do the surface integrals! Point out that this isn't possible — and that the instructions say not to.
- Student may be surprised at first when they calculate $\nabla \cdot \vec{F} = 0$, especially since they (correctly) won't think that the surface integrals will add to zero. Use this to motivate the "missing top".
- Some students incorrectly think that $d|z| = |dz|$.
- It can be tough for people to take the image and develop an equation for the sides of the net. This is healthy.

0.1.2 Subsidiary ideas

- The geometry of flux.

0.1.3 Enrichment

- The surface integrals can in fact be done — provided one adds them up prior to evaluating the integrals.
- This lab provides a good opportunity for students to visualize the flux: It's easy to see that the flux of the horizontal component of this vector field must be zero geometrically. (It's even easier to see that the vertical flux must be zero.)

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- During the wrapup (or the following lecture), draw a picture such as the one above of one of the rectangular faces, showing all 4 possible choices for $d\vec{r}_1$ and $d\vec{r}_2$ (and which is which!), and discuss the integration limits in each case.
- An alternative approach to this problem is to determine $d\vec{S}$ geometrically, compute $\vec{F} \cdot \hat{n}$ explicitly, and then do the integral using "standard" (increasing) limits. There is nothing wrong with this approach, but we would discourage the use of the $d\vec{r}$ notation here for fear of making sign errors.
- One could show students the remarkable trick for integrating e^{-x^2} from 0 to ∞, by squaring and evaluating in polar coordinates.