Student handout

Missing `/var/www/paradigms_media_2/media/activity_media/trough.png`

A fishing net S is in the shape of a triangular trough, as shown in the picture. The triangular sides are at $x = 0$ and $x = 5$, the rectangular sides are at $45°$ to the vertical, and the bottom is at $z = 0$; all lengths are measured in cm. There is no netting across the top, which is at $z = 1$. Water is draining out of the net; the motion of the water is described by the vector field $\vec{\boldsymbol{F}} = \rho\left(ae^{\kappa z^2}\,\hat{\boldsymbol{y}} - b\,\hat{\boldsymbol{z}}\right)$ where $a = 3\,\frac{\text{cm}}{\text{s}}$, $b = 5\,\frac{\text{cm}}{\text{s}}$, $\kappa = 2\,\text{cm}^{-2}$, and ρ is the (constant) density of the water in $\frac{\text{g}}{\text{cm}^3}$. The goal of this problem is to find the best way to evaluate the flux

$$\int\!\!\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}}$$

of water *down* through S.

- Set up the above surface integral, **but do not evaluate it**
  Your answer should be ready to integrate; among other things, all substitutions should be made, and you should determine the correct limits.

- **Use the Divergence Theorem** to find another way to do the problem.
  This time, complete the computation.

0.0.1 Main ideas

- Practice doing surface integrals
- The Divergence Theorem

0.0.2 Prerequisites

- Ability to do flux integrals
- Definition of divergence
- Statement of Divergence Theorem *This lab can be used prior to covering the Divergence Theorem in class with either a minimal introduction or a restatement of the last question based on the assumption that the given vector field doesn't "lose" anything going through the net.*

0.0.3 Warmup

- Perhaps a reminder about what the Divergence Theorem is.

0.0.4 Props

- whiteboards and pens
- a model of the fishing net, made from any children's building set

0.0.5 Wrapup

- Reiterate that the Divergence Theorem only applies to closed surfaces.
- Emphasize that the Divergence Theorem is one of several astonishing theorems relating what happens inside to what happens outside.
- Have several students show how they computed $d\vec{\boldsymbol{S}}$, since most likely different choices were made for $d\vec{\boldsymbol{r}}_i$ and hence the limits.

0.1 Details

0.1.1 In the Classroom

- By now the groups should be working well. Sit back and watch!
- The main thing to watch out for is whether students choose the correct signs, both for the normal vectors and the limits of integration. Reiterate that one should *always* write $d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}$; there should *never* be minus signs in this equation. The signs will come out right provided one integrates in the direction of the vectors chosen.
- Most students will realize quickly that there is no flux through the triangular sides. (encourage supporting this with geometry and a calculation).
- Some students will try to do the surface integrals! Point out that this isn't possible — and that the instructions say not to.
- Student may be surprised at first when they calculate $\nabla \cdot \vec{\boldsymbol{F}} = 0$, especially since they (correctly) won't think that the surface integrals will add to zero. Use this to motivate the "missing top".
- Some students incorrectly think that $d|z| = |dz|$.
- It can be tough for people to take the image and develop an equation for the sides of the net. This is healthy.

0.1.2 Subsidiary ideas

- The geometry of flux.

0.1.3 Enrichment

- The surface integrals can in fact be done — provided one adds them up prior to evaluating the integrals.
- This lab provides a good opportunity for students to visualize the flux: It's easy to see that the flux of the horizontal component of this vector field must be zero geometrically. (It's even easier to see that the vertical flux must be zero.)

Missing `/var/www/paradigms_media_2/media/activity_media/net.png`

- During the wrapup (or the following lecture), draw a picture such as the one above of one of the rectangular faces, showing all 4 possible choices for $d\vec{\boldsymbol{r}}_1$ and $d\vec{\boldsymbol{r}}_2$ (and which is which!), and discuss the integration limits in each case.
- An alternative approach to this problem is to determine $d\vec{\boldsymbol{S}}$ geometrically, compute $\vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}$ explicitly, and then do the integral using "standard" (increasing) limits. There is nothing wrong with this approach, but we would discourage the use of the $d\vec{\boldsymbol{r}}$ notation here for fear of making sign errors.
- One could show students the remarkable trick for integrating e^{-x^2} from 0 to ∞, by squaring and evaluating in polar coordinates.