1 Quantum States

The wavefunction is related to writing the state in the position basis (or the position "representation") by inserting a completeness relation for the position basis.:

$$\begin{aligned}
|\psi\rangle &= (1)\,|\psi\rangle \\
&= \left(\int_{\text{all space}} |x\rangle\,\langle x|\,dx\right)|\psi\rangle \\
&= \int_{\text{all space}} |x\rangle\,\langle x|\psi\rangle\,dx \\
&= \int_{\text{all space}} \underbrace{\langle x|\psi\rangle}_{\text{wavefunction}} |x\rangle\,dx \\
&= \int_{\text{all space}} \underbrace{\psi(x)}_{\text{wavefunction}} |x\rangle\,dx
\end{aligned}$$

Bra-ket Notation: $|\psi\rangle \doteq \langle x|\psi\rangle$
Wavefunction Notation: $|\psi\rangle \doteq \psi(x)$
The dot equals sign indicates that the right hand side is an expression of the state in a specific but unwritten basis.

1.1 The Hermitian Adjoint of a Quantum State:

$$\begin{aligned}
\langle\psi| &= (|\psi\rangle)^* \\
&\doteq (\langle x|\psi\rangle)^* \\
&\doteq \langle\psi|x\rangle \\
\langle\psi| &\doteq \psi^*(x)
\end{aligned}$$

Bra-ket Notation: $\langle\psi| \doteq \langle\psi|x\rangle$
Wavefunction Notation: $\langle\psi| \doteq \psi^*(x)$

2 Calculating a Probability Amplitude

For example, for an energy measurement: **Bra-ket Notation:**

$$c_n = \langle E_n|\psi\rangle$$

Wavefunction Notation:
Go to wavefunction notation by inserting a completeness relation for the position basis.

$$\begin{aligned}
c_n &= \langle E_n|\, 1\, |\psi\rangle \\
&= \langle E_n| \left(\int_{\text{all space}} |x\rangle \langle x|\, dx \right) |\psi\rangle \\
&= \int_{\text{all space}} \langle E_n|x\rangle \langle x|\psi\rangle\, dx \\
&= \int_{\text{all space}} E_n^*(x)\, \psi(x) dx
\end{aligned}$$

3 Calculating a Probability

Bra-ket Notation:

$$\begin{aligned}
\mathcal{P}(E_n) &= |c_n|^2 \\
&= |\, \langle E_n|\psi\rangle\, |^2
\end{aligned}$$

Wavefunction Notation:
Using the result for the probability amplitude in wavefunction notation:

$$\begin{aligned}
\mathcal{P}(E_n) &= |c_n|^2 \\
&= \left| \int_{\text{all space}} E_n^*(x)\, \psi(x) dx \right|^2
\end{aligned}$$

Notice the norm square is outside of the integral!

4 Normalizing a State

Bra-ket Notation:

$$1 = \langle \psi|\psi\rangle$$

Wavefunction Notation:
Go to wavefunction notation by inserting a completeness relation for the position basis.

$$
\begin{aligned}
1 &= \langle\psi|\, 1\, |\psi\rangle \\
&= \langle\psi| \left(\int_{\text{all space}} |x\rangle \langle x|\, dx \right) |\psi\rangle \\
&= \int_{\text{all space}} \langle\psi|x\rangle \langle x|\psi\rangle\, dx \\
&= \int_{\text{all space}} \psi^*(x)\, \psi(x) dx \\
&= \int_{\text{all space}} |\psi(x)|^2 dx
\end{aligned}
$$

Notice norm squared inside the integral!

5 Finding the Probability for a Range of Values

Bra-ket Notation:

$$\mathcal{P}(a \leq n \leq b) = \sum_{n=a}^{b} |c_n|^2$$

Wavefunction Notation:
Reinterpret $|\psi(x)|^2$ as a probability density:

$$\mathcal{P}(a \leq x \leq b) = \int_a^b |\psi(x)|^2 dx$$

Notice norm squared inside the integral!