Begin with Schrödinger's Equation:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle \tag{1}$$

If we know how the Hamiltonian acts on a state, we can use Schrodinger's equation to find out how it evolves in time.

Let's do this for the case of the particle on a ring (c.f. McIntyre 3.1 for a spin 1/2 system). Write out the arbitrary state as a superposition of energy eigenstates, putting the time dependence we are trying to find in the expansion coefficient.

$$i\hbar \frac{d}{dt} \sum_{m=-\infty}^{\infty} c_m(t) |m\rangle = \hat{H} \sum_{m=-\infty}^{\infty} c_m(t) |m\rangle \tag{2}$$

We are writing our state in this basis because we know what the Hamiltonian does to these kets $|m\rangle$ via the energy eigenvalue equation.

$$\hat{H} |m\rangle = E_m |m\rangle \tag{3}$$

Using this result, we can calculate:

$$\begin{aligned}
i\hbar \frac{d}{dt} \sum_{m=-\infty}^{\infty} c_m(t) |m\rangle &= \hat{H} \sum_{m=-\infty}^{\infty} c_m(t) |m\rangle && (4)\\
&= \sum_{m=-\infty}^{\infty} c_m(t) \hat{H} |m\rangle && (5)\\
&= \sum_{m=-\infty}^{\infty} E_m c_m(t) |m\rangle && (6)\\
&= \sum_{m=-\infty}^{\infty} \left(\frac{\hbar^2}{2I} m^2\right) c_m(t) |m\rangle && (7)
\end{aligned}$$

The energies E_m are the energies for the physical system we care about, i.e. a particle on a ring. If we had a different system, a different Hamiltonian, the structure of this equation would be similar but the values of energy in Eqn. (7) would be different.

Because we are solving for the time-dependent coefficients, let's get rid of the sum using an inner product with an arbitratry state $\langle k|$ by using the orthonormality condition.

$$\begin{aligned}
\langle k| i\hbar \frac{d}{dt} \sum_{m=-\infty}^{\infty} c_m(t) |m\rangle &= \langle k| \sum_{m=-\infty}^{\infty} E_m c_m(t) |m\rangle && (8)\\
i\hbar \frac{d}{dt} \sum_{m=-\infty}^{\infty} c_m(t) \langle k|m\rangle &= \sum_{m=-\infty}^{\infty} E_m c_m(t) \langle k|m\rangle && (9)\\
i\hbar \frac{d}{dt} \sum_{m=-\infty}^{\infty} c_m(t) \delta_{km} &= \sum_{m=-\infty}^{\infty} E_m c_m(t) \delta_{km} && (10)\\
i\hbar \frac{d}{dt} c_k(t) &= E_k c_k(t) && (11)\\
& && (12)
\end{aligned}$$

We now have a first-order ODE for the unknown time-dependent coefficients.

$$\frac{d}{dt}c_k(t) = -\frac{iE_k}{\hbar}c_k(t) \tag{13}$$

Because this ODE is linear with constant coefficients, the solution is an exponential.

$$c_k(t) = Ae^{-\frac{iE_k}{\hbar}t} \tag{14}$$

The constant factor A is what we have been calling the time-independent version of c_k up to this point; it is the probability amplitude of the eigenstate it accompanies at $t = 0$. You can plug in $t = 0$ to help verify this.

The solution is:

$$|\Psi(t)\rangle = \sum_{m=-\infty}^{\infty} e^{-\frac{iE_m}{\hbar}t} c_m |m\rangle \tag{15}$$

$$= \sum_{m=-\infty}^{\infty} e^{-\frac{i\hbar}{2I}m^2 t} c_m |m\rangle \tag{16}$$

Here are some important features of the solution:

- We can only put time evolution phases on states that are written in the energy basis;
- The time dependent phase is pure imaginary; it does not effect the normalization of the state;
- Each eigenstate gets its own time-dependent phase that includes the energy of that eigenstate.

It would be valuable for you to go through this derivation and see which parts depend on the particular system we are using (the quantum ring) and which parts are generic to any solution of Schrödinger's Equation.