We will show that the components of the angular momentum operator $\vec{L}$, written in differential operator form in rectangular components, satisfy the commutation relations:

$$[L_x, L_y] = +i\hbar L_z \qquad \text{(and cyclic permutations)} \tag{1}$$

First calculate the components of angular momentum classically:

$$\vec{L} = \vec{r} \times \vec{p} \tag{2}$$
$$= (x\hat{x} + y\hat{y} + z\hat{z}) \times (p_x\hat{x} + p_y\hat{y} + p_z\hat{z}) \tag{3}$$
$$= (yp_z - zp_y)\hat{x} + (zp_x - xp_z)\hat{y} + (xp_y - yp_x)\hat{z} \tag{4}$$

Making the standard quantum substitutions,

$$p_x \to -i\hbar\partial_x \tag{5}$$
$$p_y \to -i\hbar\partial_y \tag{6}$$
$$p_z \to -i\hbar\partial_z \tag{7}$$
$$\tag{8}$$

we obtain the following operators for the components of angular momentum:

$$\hat{L}_x = -i\hbar(y\partial_z - z\partial_y) \tag{9}$$
$$\hat{L}_y = -i\hbar(z\partial_x - x\partial_z) \tag{10}$$
$$\hat{L}_z = -i\hbar(x\partial_y - y\partial_x) \tag{11}$$
$$\tag{12}$$

To see the role of the product rule in the commutation relations, it is helpful to give the partial derivatives an arbitrary function ψ to act on.

$$\left[\hat{L}_x, \hat{L}_y\right]\psi = \left[-i\hbar(y\partial_z - z\partial_y), -i\hbar(z\partial_x - x\partial_z)\right]\psi \tag{13}$$
$$= -\hbar^2\left\{(y\partial_z - z\partial_y)(z\partial_x - x\partial_z) - (z\partial_x - x\partial_z)(y\partial_z - z\partial_y)\right\}\psi \tag{14}$$

Now, foil-like-mad. Make sure that all of the partial derivatives act on EVERYTHING to their right. Two of the terms above of the form

$$y\,\partial_z(z\,\partial_x\psi) \tag{15}$$

require a product rule:

$$y\,\partial_z(z\,\partial_x\psi) = y((\partial_z z)(\partial_x\psi) + z(\partial_x\partial_x\psi)) \tag{16}$$
$$= y\partial_x\psi + yz(\partial_x\partial_x\psi) \tag{17}$$

Continuing the calculation above, we see that all of the second derivative terms will cancel because the order of differentiation doesn't matter, leaving only the first derivative terms from the product rule.

$$
\begin{aligned}
\left[\hat{L}_x, \hat{L}_y\right] \psi &= \left[-i\hbar(y\partial_z - z\partial_y), -i\hbar(z\partial_x - x\partial_z)\right] \psi && (18)\\
&= -\hbar^2 \left\{(y\partial_z - z\partial_y)(z\partial_x - x\partial_z) - (z\partial_x - x\partial_z)(y\partial_z - z\partial_y)\right\} \psi && (19)\\
&= -\hbar^2 \left\{(y\,\partial_z(z\,\partial_x\psi) - y\,\partial_z(x\,\partial_z\psi) - z\,\partial_y(z\,\partial_x\psi) + z\,\partial_y(x\,\partial_z\psi))\right. && (20)\\
&\qquad \left. - (z\,\partial_x(y\,\partial_z\psi) - z\,\partial_x(z\,\partial_y\psi) - x\,\partial_z(y\,\partial_z\psi) + x\,\partial_z(z\,\partial_y\psi))\right\} && (21)\\
&= -\hbar^2 \left\{\left(\cancel{yz(\partial_z\partial_x\psi)} + y\partial_x\psi - \cancel{yx(\partial_z^2\psi)} - \cancel{z^2(\partial_y\partial_x\psi)} + \cancel{zx(\partial_y\partial_z\psi)}\right)\right. && (22)\\
&\qquad \left. - \left(\cancel{zy(\partial_x\partial_z\psi)} - \cancel{z^2(\partial_x\partial_y\psi)} - \cancel{xy(\partial_z^2\psi)} + \cancel{xz(\partial_z\partial_y\psi)} + x\partial_y\psi\right)\right\} && (23)\\
&= i\hbar\left(-i\hbar(-y\partial_x + x\partial_y)\psi\right) && (24)\\
&= i\hbar\hat{L}_z\,\psi && (25)
\end{aligned}
$$

The other components are cyclic permutations of this calculation.