A completeness relation is a really fancy way of writing the identity operator. You can write a completeness relation for ANY quantum system in ANY complete basis.

For spin 1/2 systems in the z-basis, we have

$$1 = \left|+\right\rangle \left\langle+\right| + \left|-\right\rangle \left\langle-\right| \tag{1}$$

$$\doteq \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \tag{2}$$

$$= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{3}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{4}$$

The trick is to SUM the ket/bras over a complete set of basis states.

For the quantum ring in the energy basis, we have

$$1 = \sum_{m=-\infty}^{\infty} \left|m\right\rangle \left\langle m\right| \tag{5}$$

or in the position basis, we have

$$1 = \int_0^{2\pi} \left|\phi\right\rangle \left\langle\phi\right| \, r_0 \, d\phi \tag{6}$$

1. Write the completeness relations for the hydrogen atom in both the energy and the position bases.

2. Briefly interpret the symbols in the following statements involving completeness relations for the hydrogen atom:

$$\left|\psi\right\rangle = \left(\sum_{n=0}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} \left|n\ell m\right\rangle \left\langle n\ell m\right|\right)\left|\psi\right\rangle \tag{7}$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} \left|n\ell m\right\rangle c_{n\ell m} \tag{8}$$

$$\left|\psi\right\rangle = \left(\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left|r\,\theta\,\phi\right\rangle \left\langle r\,\theta\,\phi\right| \, r^2 \sin\theta \, dr\, d\theta\, d\phi\right)\left|\psi\right\rangle \tag{9}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left|r\,\theta\,\phi\right\rangle \left\langle r\,\theta\,\phi|\psi\right\rangle \, r^2 \sin\theta \, dr\, d\theta\, d\phi \tag{10}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left|r\,\theta\,\phi\right\rangle \psi(r,\theta,\phi) \, r^2 \sin\theta \, dr\, d\theta\, d\phi \tag{11}$$

$$\tag{12}$$

3. Use the hydrogen atom completeness relations to find new formulas for the following expressions.

When would you use these formulas?

$$
\begin{aligned}
c_{n\ell m} &= \langle n\ell m|\psi\rangle \\
&=? \\
\psi(r,\theta,\phi) &= \langle r\,\theta\,\phi|\psi\rangle \\
&=?
\end{aligned}
$$