Student handout

Electrostatic Potential from Two Charges

Find a third order approximation to the electrostatic potential $V(\vec{r})$ for one of the following physical situations.

1. Two charges $+Q$ and $+Q$ are placed on a line at $z' = D$ and $z'' = -D$ respectively.
   a) On the x-axis for $|x| \ll D$?
   b) On the z-axis for $|z| \ll D$?
   c) On the x-axis for $|x| \gg D$?
   d) On the z-axis for $|z| \gg D$?
2. Two charges $+Q$ and $-Q$ are placed on a line at $z' = +D$ and $z'' = -D$ respectively.
   a) On the x-axis for $|x| \ll D$?
   b) On the z-axis for $|z| \ll D$?
   c) On the x-axis for $|x| \gg D$?
   d) On the z-axis for and $|z| \gg D$?

Work out your problem by brainstorming together on a big whiteboard and also answer the following questions:

- For what values of $\vec{r}$ does your series converge?
- For what values of $\vec{r}$ is your approximation a good one?
- Which direction would a test charge move under the influence of this electric potential?

If your group gets done early, go on to another problem. The fourth problem in each set is the most challenging, and the most interesting.

1 Instructor's Guide

1.1 Introduction

Students typically know the iconic formula for the electrostatic potential of a point charge $V = \frac{kq}{r}$. We begin this activity with a short lecture/discussion that generalizes this formula in a coordinate independent way to the situation where the source is moved away from the origin to the point $\vec{r}'$, $V(\vec{r}) = \frac{kq}{|\vec{r}-\vec{r}'|}$. (A nice warm-up (SWBQ) to lead off the discussion:

Introductory SWBQ Prompt: "Write down the electrostatic potential everywhere in space due to a point charge that is not at the origin." The lecture should also review the superposition principle.

$$V(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|}$$

This general, coordinate-independent formula should be left on the board for students to consult as they do this activity.

1.2 Student Conversations

- Note: two of the eight cases on the worksheet are trivial (the potential on the y-axis is zero for the $+Q$ and $-Q$ situation). Once these groups have established the correct answer and can justify it, they should be directed to work on one of the other six questions.

- In the first part of this activity, students will create an expression such as $V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|D-z|} + \frac{1}{|D+z|}\right)$; each situation has a slightly different formula. A few groups may have trouble coordinatizing $|\vec{r} - \vec{r}'|$ into an expression in rectangular coordinates, but because the coordinate system is set up for them, most students are successful with this part fairly quickly.

- In the second part of the activity, students are asked to take the equation above and develop a 4th order power series expansion. About 20 minutes will be needed for this portion of the activity. Almost all students will struggle with creating the power series. Although our students have some experience with power series from mathematics courses, they are unlikely to have seen the common physics strategy of substituting into known series by rewriting into an expression in terms of dimensionless parameters and then using a known power series. The groups may need lots of guidance to find this strategy.

- If students have been exposed to Taylor's theorem $f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} +$, they will probably first attempt to apply this basic formula to this situation. Successive derivatives will rapidly lead to an algebraic mess. In general, we let students "get stuck" at this stage for only a few minutes before suggesting that they try a known power series expansion. We don't tell them which one, but they rapidly rule out formulas for trigonometric functions and other functions that clearly don't apply.

- Once students are aware that $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + ...$ is the expansion they need to be using, they still face a substantial challenge. It may not be immediately obvious to them how an expression such as $\frac{1}{|z-D|}$ can be transformed to the form $(1+z)^p$. Simply giving students the answer at this point will defeat most of the learning possibilities of this activity. Students may need some time just to recognize that $p = -1$; they may need much more time to determine if z or D is the smaller quantity and recognize that by factoring out the larger quantity they can have an expression that starts looking like $(1+z)^p$, with $z = \frac{x}{D}$ (or $\frac{D}{x}$ or) and $p = -1$.

- MANY students make algebraic errors such as incorrectly factoring out D or x. These should be brought to their attention quickly. Students may also have trouble dealing with negative exponents or with the absolute value sign.

- Students who are having trouble figuring out what parameter to expand in can be asked what quantity is small. Then they can be asked what that means — small with respect to what? This should eventually guide them to a ratio — which is small with respect to 1, as required.

- Many students are likely to treat this as a two-dimensional case from the start, ignoring the z axis entirely. Look for expressions like

$$V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{\sqrt{(x-x_i)^2 + (z-z_i)^2}}.$$

  Encourage them to think in three dimensions.

- Point out that one should NEVER expand a series in the denominator. Use a negative exponent instead. It can be surprising how many upper-division physics students still have foundational problems with negative exponents.

- **Laurent Series** For the cases that point where the potential is being evaluated is far from the source, the series that the students will find is technically a Laurent series, not a power series (the series involves INVERSE powers of the position of the detector). Nevertheless, they will be using a power series formula where the expansion parameter

$$\frac{\text{distance from the origin to the source}}{\text{distance from the origin to the detector}}$$

  is small, so their formula is valid. It is not necessary to make a big deal out of the difference between power and Laurent series, but savvy students may have questions.

- Some students may wonder why we care about the power series expansion of an expression we have in closed form? Answer: Because in applications we won't know the closed form.

1.3 Wrap-up

If time allows, each of the eight groups should have an opportunity to present their results to the class. The instructor should encourage students to compare and contrast the results for the eight situations. This should include careful attention to:

- whether the situation is symmetric or anti-symmetric and how this relates to whether the power series is odd or even;
- whether the terms of the series get successively smaller
- whether the answers "make sense" given the physical situation and what they tell you about how the field changes along the given axis.

End with a discussion that extracts from the different examples the overall method:

- Start with the iconic equation;
- Use what you know about the coordinate system and positions of objects to coordinatize $|\vec{r} - \vec{r}'|$
- Factor out the parameter that is large and use a memorized series for $(1 + z)^p$,