The Magnetic Vector Potential Due to a Spinning Ring of Charge

1. Use the superposition principle for the magnetic vector potential due to a continuous current distribution:
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}'(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau', \tag{1}$$
to find the magnetic vector potential everywhere in space due to a spinning charged ring with radius R, total charge Q, and period T.
2. Evaluate your expression for the special case that $\vec{r}$ is on the z-axis.
3. Evaluate your expression for the special case that $\vec{r}$ is on the x-axis.
4. Find a series expansion for the electrostatic potential at these special locations:
   a) Near the center of the ring, in the plane of the ring;
   b) Near the center of the ring, on the axis of the ring;
   c) Far from the ring on the axis of symmetry;
   d) Far from the ring, in the plane of the ring.

Solution

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int\limits_{\text{ring}} \frac{\vec{I}(\vec{r}')\,|d\vec{r}'|}{|\vec{r} - \vec{r}'|} \tag{2}$$

where $\vec{r}$ denotes the position in space at which the magnetic vector potential is measured and $\vec{r}'$ denotes the position of the current segment.

For the current

$$\begin{aligned}
\vec{I}(\vec{r}') &= \lambda_0 \, \vec{v} &(3)\\
&= \frac{Q}{2\pi R} \frac{2\pi R}{T} \, \hat{\phi}' &(4)\\
&= \frac{Q}{T} \, \hat{\phi}' &(5)\\
&= \frac{Q}{T} (-\sin\phi' \hat{x} + \cos\phi' \hat{y}) &(6)
\end{aligned}$$

Draw a picture to find the direction of $\hat{\phi}'$ or see this simulation.

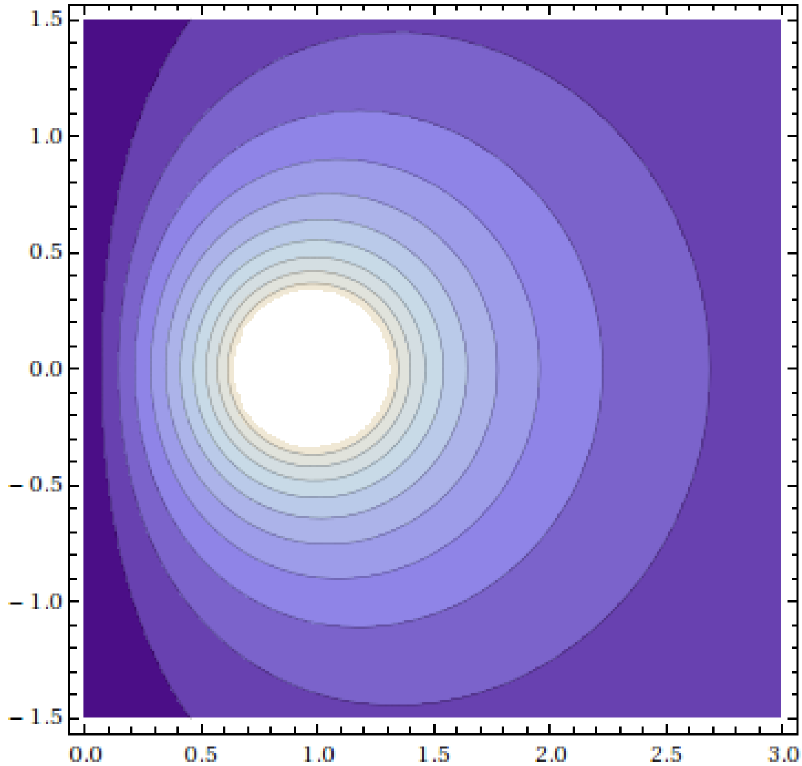
In cylindrical coordinates on the ring, $|d\vec{r}'| = R\, d\phi'$, and, as discussed in previous solutions,

$$|\vec{r} - \vec{r}'| = \sqrt{s^2 - 2sR\cos(\phi - \phi') + R^2 + z^2} \tag{7}$$

Thus

$$\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \int\limits_0^{2\pi} \frac{Q}{T} \frac{(-\sin\phi' \hat{x} + \cos\phi' \hat{y})}{\sqrt{s^2 - 2sR\cos(\phi - \phi') + R^2 + z^2}} \, R d\phi' &(8)\\
&= \frac{\mu_0}{4\pi} \frac{QR}{T} \int\limits_0^{2\pi} \frac{(-\sin\phi' \hat{x} + \cos\phi' \hat{y})}{\sqrt{s^2 - 2sR\cos(\phi - \phi') + R^2 + z^2}} \, d\phi' &(9)
\end{aligned}$$

To visualize this solution, it is sufficient to plot the value of the potential in a plane of constant ϕ due to the cylindrical symmetry of the answer. This plot is the *magnitude* of the vector field. The direction of the vector field is into the page.



The z axis

For points on the z axis, $s = 0$ and the integral simplifies to

$$\vec{A}(0,0,z) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int_0^{2\pi}\frac{(-\sin\phi'\hat{x} + \cos\phi'\hat{y})}{\sqrt{R^2+z^2}}\,d\phi' \tag{10}$$

The denominator is constant and pulls out of the integral. Doing the integral of just sine or cosine results in

$$\vec{A}(0,0,z) = 0 \tag{11}$$

The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$\vec{A}(x,0,0) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int_0^{2\pi}\frac{(-\sin\phi'\hat{x} + \cos\phi'\hat{y})}{\sqrt{x^2 - 2xR\cos\phi' + R^2}}\,d\phi' \tag{12}$$

This results in a very similar situation as the case for electric field on the x axis, except that now we will address the $\hat{x}$ component instead of the $\hat{y}$ component. Using the same process we let $u = x^2 - 2xR\cos\phi' + R^2$, then $du = 2xR\sin\phi' d\phi'$, and for the $\hat{x}$ component the integral becomes

$$A_x(x,0,0) = \frac{\mu_0}{4\pi}\frac{QR}{T}\left(\frac{-\hat{x}}{2xR}\right)\int_0^{2\pi}\frac{1}{u^{1/2}}\,du \tag{13}$$

Doing the integral, we find

$$A_x(x,0,0) = 0 \tag{14}$$

Thus the $\hat{x}$ component disappears and we are left with an elliptic integral with only a $\hat{y}$ component

$$\vec{A}(x,0,0) = \frac{\mu_0}{4\pi}\frac{QR}{T}\,\hat{y}\int_0^{2\pi}\frac{\cos\phi'}{\sqrt{x^2 - 2xR\cos\phi' + R^2}}\,d\phi' \tag{15}$$

Then we can use symmetry to argue that $\hat{y}$ on the x-axis becomes $\hat{\phi}$ anywhere on the x, y-plane, so that

$$\vec{A}(s,0,0) = \frac{\mu_0}{4\pi}\frac{QR}{T}\,\hat{\phi}\int_0^{2\pi} \frac{\cos\phi'}{\sqrt{s^2 - 2sR\cos\phi' + R^2}}\,d\phi' \tag{16}$$