The Magnetic Field Due to a Spinning Ring of Charge

1. Use the Biot-Savart law
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d\tau'$$
to find the magnetic field everywhere in space due to a spinning charged ring with radius R, total charge Q, and period T.

2. Evaluate your expression for the special case that $\vec{r}$ is on the z-axis.

3. Evaluate your expression for the special case that $\vec{r}$ is on the x-axis.

4. Find a series expansion for the electrostatic potential at these special locations:

   a) Near the center of the ring, in the plane of the ring;
   b) Near the center of the ring, on the axis of the ring;
   c) Far from the ring on the axis of symmetry;
   d) Far from the ring, in the plane of the ring.

Solution

1.
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\limits_{\text{ring}} \frac{\vec{I}(\vec{r}') \times (\vec{r}-\vec{r}')\,|dr'|}{|\vec{r}-\vec{r}'|^3} \tag{1}$$

where $\vec{r}$ denotes the position in space at which the magnetic field is measured and $\vec{r}'$ denotes the position of the current segment. As described in previous solutions,

$$|dr'| = R\,d\phi' \tag{2}$$
$$\vec{I}(\vec{r}') = \frac{Q}{2\pi R}\frac{2\pi R}{T}\hat{\phi}' \tag{3}$$
$$= \frac{Q}{T}(-\sin\phi'\hat{x} + \cos\phi'\hat{y}) \tag{4}$$
$$\vec{r}-\vec{r}' = (s\cos\phi - R\cos\phi')\hat{x} + (s\sin\phi - R\sin\phi')\hat{y} + (z-z')\hat{z} \tag{5}$$
$$|\vec{r}-\vec{r}'| = \sqrt{s^2 - 2sR\cos(\phi-\phi') + R^2 + z^2} \tag{6}$$

Thus $\vec{B}(\vec{r}) =$

$$\frac{\mu_0}{4\pi}\frac{QR}{T}\int_0^{2\pi} \frac{(-\sin\phi'\hat{x} + \cos\phi'\hat{y}) \times [(s\cos\phi - R\cos\phi')\hat{x} + (s\sin\phi - R\sin\phi')\hat{y} + z\hat{z}]}{\left(\sqrt{s^2 - 2sR\cos(\phi-\phi') + R^2 + z^2}\right)^3}\,d\phi' \tag{7}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int_0^{2\pi} \frac{(z\cos\phi'\hat{x} + z\sin\phi'\hat{y} + [R - s\cos(\phi-\phi')]\hat{z})}{\left(\sqrt{s^2 - 2sR\cos(\phi-\phi') + R^2 + z^2}\right)^3}\,d\phi' \tag{8}$$

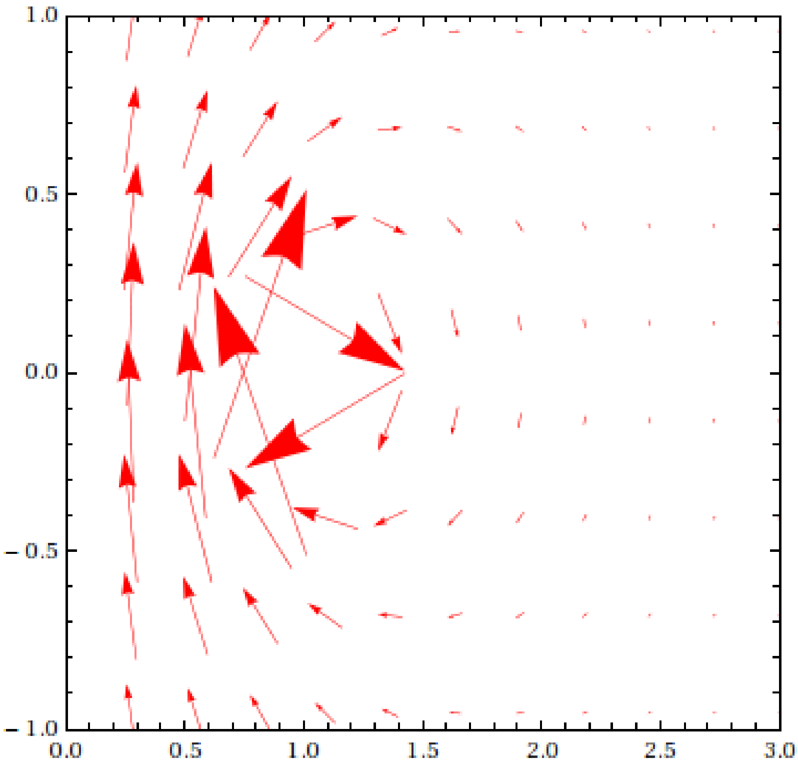
2. On the z-axis, we have $s = 0$, so the previous equation reduces to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int_0^{2\pi} \frac{(z\cos\phi'\hat{x} + z\sin\phi'\hat{y} + R\hat{z})}{\left(\sqrt{R^2+z^2}\right)^3}\,d\phi' \tag{9}$$

The $\hat{x}$ and $\hat{y}$ terms integrate the functions $\cos\phi'$ and $\sin\phi'$ over a full period, so these integrals are zero. The $\hat{z}$ integral is the integral of a constant, so the final answer is

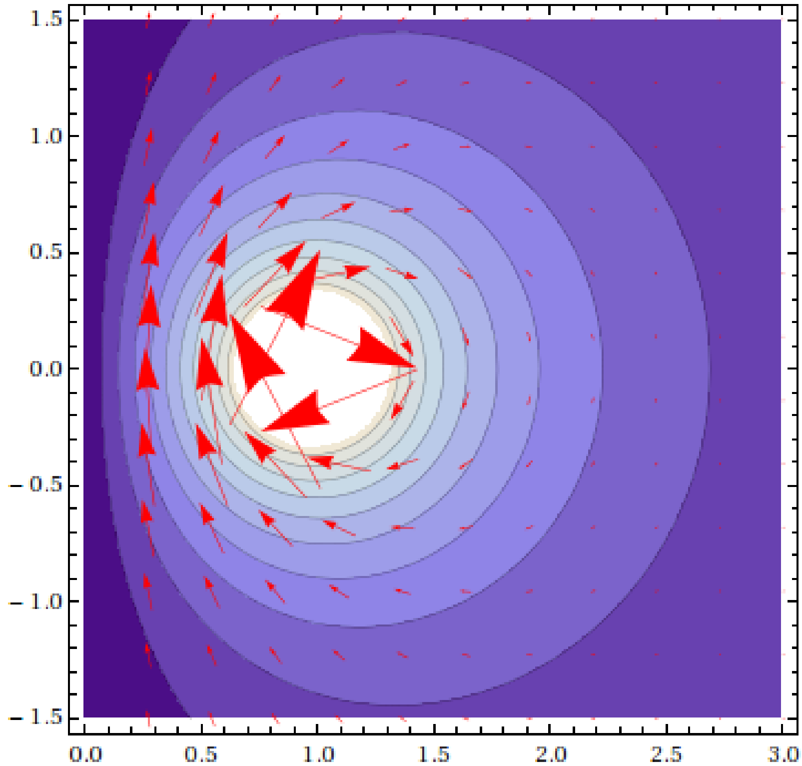
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\frac{2\pi R}{(R^2+z^2)^{3/2}}\,\hat{z} \tag{10}$$

To visualize this solution, it is sufficient to plot the value of the field in a plane of constant ϕ due to the cylindrical symmetry of the answer.



If we plot the magnetic field on top of the (magnitude of the) magnetic vector potential, we can see that the first is the curl of the second

$$\vec{B} = \vec{\nabla}\times\vec{A}$$



The z axis

For points on the z axis, $s = 0$ and ϕ can without loss of generality taken as zero. Thus, the integral simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int_0^{2\pi}\frac{[z\cos\phi'\hat{x} + z\sin\phi'\hat{y} + R\hat{z}]}{\left(\sqrt{R^2+z^2}\right)^3}\,d\phi' \tag{11}$$

Doing the integral results in

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\frac{2\pi R\,\hat{z}}{\left(\sqrt{R^2+z^2}\right)^3} \tag{12}$$

The x axis

For points on the x axis, $z = 0$ and $\phi = 0$. Because $z = 0$ the $\hat{x}$ and $\hat{y}$ components disappear and the integral simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR}{T}\int_0^{2\pi}\frac{(R - s\cos\phi')\hat{z}}{\left(\sqrt{s^2 - 2sR\cos\phi' + R^2}\right)^3}\,d\phi' \tag{13}$$