The Electrostatic Potential Due to a Ring of Charge

1. Use the superposition principle for the electrostatic potential due to a continuous charge distribution:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho'(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d\tau', \tag{1}$$

   to find the electrostatic potential everywhere in space due to a uniformly charged ring with radius R and total charge Q.

   Check with a teaching team member before moving on to subsequent parts below.

2. Evaluate your expression for the special case of the potential on the z-axis.

3. Evaluate your expression for the special case of the potential on the x-axis.

4. Find a series expansion for the electrostatic potential in these special regions:

   a) Near the center of the ring, in the plane of the ring;
   b) Near the center of the ring, on the axis of the ring;
   c) Far from the ring on the axis of symmetry;
   d) Far from the ring, in the plane of the ring.

Solution

1. The formula for the electrostatic potential due to a ring of charge at an **arbitrary** point in space is an eliptic integral, which a computer algebra system such as *Mathematica* is capable of evaluating point by point. The formula is:

$$V(s, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} \int_0^{2\pi} \frac{R\, d\phi'}{\sqrt{s^2 + R^2 - 2Rs\cos(\phi - \phi') + z^2}}$$

2. Along the z-axis: It may worry you that the value of ϕ, and therefore also the value of $\cos(\phi - \phi')$, is undefined on the z-axis. Fortunately, since the value of s is zero there, and the cosine function is bounded even though it is multiple-valued, so it is certainly finite. Therefore, the troublesome term in the denominator of the integrand is zero and we find:

$$V(0, \text{anything}, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} \int_0^{2\pi} \frac{R\, d\phi'}{\sqrt{R^2 + z^2}} \tag{2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} \frac{R}{\sqrt{R^2 + z^2}} \int_0^{2\pi} d\phi' \tag{3}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{R^2 + z^2}} \tag{4}$$

   Notice that this expression has the correct dimensions. For $z >> R$, so that we can neglect the R^2 term in the denominator, we see that the potential is approximately the potential of a point charge Q at the origin, as if the entire ring is squashed to a point.

3. Along the x-axis: We have $z = 0$ and $s = x$. The expression for the potential doesn't simplify much and we are still unable to complete the integral:

$$V(x, 0, 0) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} \int_0^{2\pi} \frac{R\, d\phi'}{\sqrt{x^2 + R^2 - 2Rx\cos(\phi - \phi')}}$$

4. *** Add this solution To visualize the complete solution, due to the cylindrical symmetry of the answer, it is sufficient to plot an equipotential diagram of the value of the potential in a plane of constant ϕ.