Representations of the Infinite Square Well

Consider three particles of mass m which are each in an infinite square well potential at $0 < x < L$. The energy eigenstates of the infinite square well are:

$$E_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

with energies $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$
The particles are initially in the states, respectively:

$$\begin{aligned}
|\psi_a(0)\rangle &= A\Big[2i\,|E_4\rangle - 3\,|E_{10}\rangle\Big] \\
\psi_b(x,0) &= B\left[i\sqrt{\frac{8}{L}}\sin\left(\frac{4\pi x}{L}\right) - \sqrt{\frac{18}{L}}\sin\left(\frac{10\pi x}{L}\right)\right] \\
\psi_c(x,0) &= Cx(x-L)
\end{aligned}$$

For each particle:

1. Determine the normalization constant.

Solution

a) Particle a:

$$1 = \langle\psi_a|\psi_a\rangle$$
$$\rightarrow A = \frac{1}{\sqrt{14}}$$

b) Particle b:

$$1 = \int_0^L |\psi_b(x)|^2 dx$$
$$\rightarrow A = \frac{1}{\sqrt{14}}$$

c) Particle c:

$$1 = \int_0^L |\psi_b(x)|^2 dx$$
$$= \int_0^L |C|^2 x^2 (x-L)^2 dx$$
$$\rightarrow C = \sqrt{\frac{30}{L^5}}$$

The integral can be done by multiplying and getting a polynomial.

2. At $t = 0$ what is the probability of measuring the energy of the particle to be $\frac{8\pi^2\hbar^2}{mL^2}$?

Solution

a) Particle a:

$$\mathcal{P}\Big(E = \frac{16\pi^2\hbar^2}{2mL^2}\Big) = \Big|\, \langle E_4|\psi_a(0)\rangle \,\Big|^2$$
$$= \Big|\, \langle\phi_4|\, \frac{1}{\sqrt{14}}\left[|\phi_1\rangle + 2i|\phi_4\rangle - 3|\phi_{10}\rangle\right] \Big|^2$$
$$= \frac{2}{7}$$

b) Particle b: It's easiest to translate to Dirac notation, and then the solution is exactly the same as for Particle a. If I insist on doing wavefunction notation:

$$\mathcal{P}\Big(E = \frac{16\pi^2\hbar^2}{2mL^2}\Big) = \Big|\, \langle E_4|\psi_b(0)\rangle \,\Big|^2$$
$$= \Big| \int_0^L \sqrt{\frac{2}{L}} \sin\frac{4\pi x}{L} \frac{1}{\sqrt{14}} \Big[\sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} +$$
$$2i\sqrt{\frac{2}{L}} \sin\frac{4\pi x}{L} - \sqrt{\frac{2}{L}} \sin\frac{10\pi x}{L}\Big] dx\Big|^2$$
$$= \frac{2}{7}$$

c) Particle c:

$$\mathcal{P}\Big(E = \frac{16\pi^2\hbar^2}{2mL^2}\Big) = \Big|\, \langle E_4|\psi_c(0)\rangle \,\Big|^2$$
$$= \Big| \int_0^L \sqrt{\frac{2}{L}} \sin\frac{4\pi x}{L} \sqrt{\frac{30}{L^5}} x(x-L) dx\Big|^2$$
$$= 0$$

3. Find state of the particle at a later time t.

Solution Stick in a complex, time-dependent phase on each term in the energy eigenstate expansion of the initial state:

a) Particle a:

$$|\psi_a(t)\rangle = \frac{1}{\sqrt{14}}\left[e^{-iE_1t/\hbar}\,|\phi_1\rangle + 2ie^{-iE_4t/\hbar}\,|\phi_4\rangle - 3e^{-iE_{10}t/\hbar}\,|\phi_{10}\rangle\right]$$
$$= \frac{1}{\sqrt{14}}\left[e^{-i\pi^2\hbar t/2mL^2}\,|\phi_1\rangle + 2ie^{-i4\pi^2\hbar t/2mL^2}\,|\phi_4\rangle - 3e^{-i100\pi^2\hbar t/2mL^2}\,|\phi_{10}\rangle\right]$$

b) Particle b:

$$\psi_b(x,t) = \frac{1}{\sqrt{14}}\left[e^{-i\pi^2\hbar t/2mL^2}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) + ie^{-i4\pi^2\hbar t/2mL^2}\sqrt{\frac{8}{L}}\sin\left(\frac{4\pi x}{L}\right) - e^{-i100\pi^2\hbar t/2mL^2}\sqrt{\frac{18}{L}}\sin\left(\frac{10\pi x}{L}\right)\right]$$

c) Particle c: First, I have to write the state as a superposition of eigenstates:

$$\psi_c(x,0) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$$

where

$$\begin{aligned} c_n &= \int_0^L \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \sqrt{\frac{30}{L^5}} x(x-L)dx \\ &= \sqrt{30}\,\frac{-2+2\cos n\pi}{n^3\pi^3} \end{aligned}$$

for integer $n > 0$. Then, stick a complex, time-dependent phase on each term:

$$\psi_c(x,t) = \sum_{n=1}^{\infty} \sqrt{30}\,\frac{-2+2\cos n\pi}{n^3\pi^3} e^{-in^2\pi^2\hbar t/2mL^2} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$$

4. What is the probability of measuring the energy of the particle to be the same value $\frac{8\pi^2\hbar^2}{mL^2}$ at a later time t?

   Solution Since the Hamiltonian does not depend on time, the probability of measuring a particular energy value doesn't change with time, so the answers are the same as before.

5. What is the probability of finding the particle to be in the first half of the well?

   Solution

   a) Particle a: This calculation cannot be done with Dirac notation, so I have to switch to wavefunction notation:

$$\begin{aligned} \mathcal{P}(0<x<L/2) &= \int_0^{L/2} \left|\psi_a(x,t)\right|^2 dx \\ &= \int_0^{L/2} \left| \frac{1}{\sqrt{14}} \left[e^{-i\pi^2\hbar t/2mL^2} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \right.\right. \\ &\quad + ie^{-i4\pi^2\hbar t/2mL^2} \sqrt{\frac{8}{L}} \sin\left(\frac{4\pi x}{L}\right) \\ &\quad \left.\left. - e^{-i100\pi^2\hbar t/2mL^2} \sqrt{\frac{18}{L}} \sin\left(\frac{10\pi x}{L}\right) \right] \right|^2 dx \end{aligned}$$

   b) Particle b: Same as for Particle a

   c) Particle c:

$$\begin{aligned} \mathcal{P}(0<x<L/2) &= \int_0^{L/2} \left|\psi_c(x,t)\right|^2 dx \\ &= \int_0^{L/2} \left| \sum_{n=1}^{\infty} \sqrt{30}\,\frac{-2+2\cos n\pi}{n^3\pi^3} e^{-in^2\pi^2\hbar t/2mL^2} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \right|^2 dx \end{aligned}$$