Block Sliding Down a Frictionless Wedge

(Taylor Example 7.5) Consider a block with mass m sliding frictionlessly down an wedge with mass M that makes an angle α. The wedge itself slides frictionlessly across a horizontal floor near the surface of Earth. The block is released from the top of the wedge, with both objects initially at rest.

If length of the sloping face is d, how long does the block take to reach the bottom?

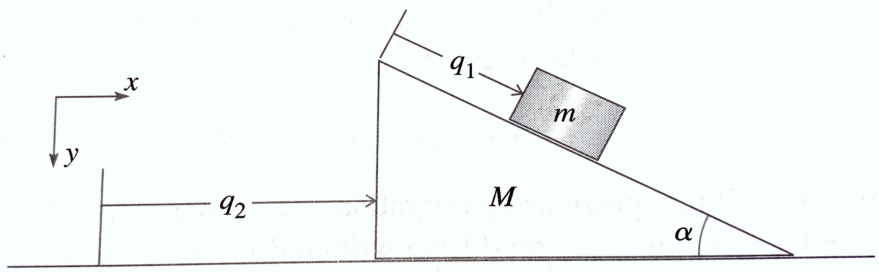

Figure 7.8 A block of mass m slides down a wedge of mass M, which is free to slide over the horizontal table.

Solution Since the generalized coordinates are defined in the picture, I'll start by writing down the Lagrangian:

$$\mathcal{L} = KE - PE$$

$$KE = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2$$

Finding v_m is tricky, but here are two ways to do it:

<u>Method 1:</u> Use the Cartesian basis vectors.

$$\vec{r_m} = (q_2 + q_1\cos\alpha)\hat{x} + q_1\sin\alpha\hat{y}$$

$$\vec{v}_m = \frac{d}{dt}\vec{r} = (\dot{q}_2 + \dot{q}_1\cos\alpha)\hat{x} + \dot{q}_1\sin\alpha\hat{y}$$

$$\begin{aligned} v_m^2 &= \vec{v}\cdot\vec{v} \\ &= (\dot{q}_2 + \dot{q}_1\cos\alpha)^2 + (\dot{q}_1\sin\alpha)^2 \\ &= \dot{q}_2^2 + \dot{q}_1^2\cos^2\alpha + 2\dot{q}_1\dot{q}_2\cos\alpha + \dot{q}_1^2\sin^2\alpha \\ &= \dot{q}_2^2 + \dot{q}_1^2(\cancelto{1}{\cos^2\alpha + \sin^2\alpha}) + 2\dot{q}_1\dot{q}_2\cos\alpha \\ v_m^2 &= \dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha \end{aligned}$$

Method 2: Use non-orthogonal basis vectors.

$$\vec{r_m} = q_2\hat{q}_2 + q_1\hat{q}_1$$

where $\hat{q}_2$ and $\hat{q}_1$ are normal but not orthogonal:

$$\begin{aligned} \hat{q}_2 \cdot \hat{q}_2 &= 1 \\ \hat{q}_1 \cdot \hat{q}_1 &= 1 \\ \hat{q}_2 \cdot \hat{q}_1 &= \cos\alpha \end{aligned}$$

Now take the derivative to find the velocity:

$$\begin{aligned} \vec{v}_m &= \frac{d}{dt}\vec{r} = \dot{q}_2\hat{q}_2 + \dot{q}_1\hat{q}_1 \\ v_m^2 &= \vec{v} \cdot \vec{v} \\ &= (\dot{q}_2\hat{q}_2 + \dot{q}_1\hat{q}_1) \cdot (\dot{q}_2\hat{q}_2 + \dot{q}_1\hat{q}_1) \end{aligned}$$

Multiplying (or FOILing) I get:

$$\begin{aligned} v_m^2 &= \dot{q}_2^2\cancelto{1}{(\hat{q}_2 \cdot \hat{q}_2)} + \dot{q}_2\dot{q}_1(\hat{q}_2 \cdot \hat{q}_1) + \dot{q}_1\dot{q}_2(\hat{q}_1 \cdot \hat{q}_2) + \dot{q}_1^2\cancelto{1}{(\hat{q}_1 \cdot \hat{q}_1)} \\ v_m^2 &= \dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha \end{aligned}$$

Finding v_M is more straightforward because the coordinate q_2 completely describes its position:

$$v_M^2 = \dot{q}_2^2$$

So, the kinetic energy is:

$$KE = \frac{1}{2}m\Big(\dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\Big) + \frac{1}{2}M\dot{q}_2^2$$

The potential energy is near Earth potential energy and only the box changes potential energy:

$$PE = -mgq_1\sin\alpha$$

So, the Lagrangian function is:

$$\begin{aligned} \mathcal{L} &= KE - PE \\ \mathcal{L} &= \frac{1}{2}m\Big(\dot{q}_2^2 + \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\Big) + \frac{1}{2}M\dot{q}_2^2 + mgq_1\sin\alpha \end{aligned}$$

Now, apply the Euler-Lagrange Equation for each coordinate:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}$$
$$mg\sin\alpha = \frac{d}{dt}(m\dot{q}_1 + m\dot{q}_2\cos\alpha)$$
$$\therefore \quad g\sin\alpha = \ddot{q}_1 + \ddot{q}_2\cos\alpha$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}$$
$$0 = \frac{d}{dt}(m\dot{q}_2 + m\dot{q}_1\cos\alpha + M\dot{q}_2)$$
$$0 = (m+M)\ddot{q}_2 + m\ddot{q}_1\cos\alpha$$

$$\therefore \quad \ddot{q}_2 = \frac{m\cos\alpha}{m+M}\ddot{q}_1$$

These are coupled 2nd order differential equations, but they are linear and I can decouple them by substitution:

$$g\sin\alpha = \ddot{q}_1 + \left(\frac{m\cos\alpha}{m+M}\ddot{q}_1\right)\cos\alpha$$
$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{m+M}}$$
$$= \frac{g\sin\alpha(m+M)}{m+M-m\cos^2\alpha}$$
$$\ddot{q}_1 = \frac{g\sin\alpha(m+M)}{M+m(1-\cos^2\alpha)} \checkmark$$

$$\ddot{q}_2 = \left(\frac{m\cos\alpha}{m+M}\right)\left(\frac{g\sin\alpha(m+M)}{M+m(1-\cos^2\alpha)}\right)$$
$$\ddot{q}_2 = \frac{mg\sin\alpha\cos\alpha}{M+m(1-\cos^2\alpha)} \checkmark$$

Do some evaluative sensemaking (Does my answer make sense?)

- Dimensions - both accelerations have dimensions of acceleration (or Length/Time^2) ✓

- Functional Behavior - if I think about the system as a whole, the net force on the system is due to the acceleration due to gravity and the normal force from the floor on the wedge. These forces are constant - therefore, I expect the accelerations to be constant - and they are!

  Are there any momenta conserved? Yes! Since the net force on the system is vertical, I expect the horizontal momentum to be conserved. q_2 is missing from the Lagrangian, so the momentum $\partial\mathcal{L}/\partial\dot{q}_2$ is constant is time, i.e., conserved. This looks like:

$$M\dot{q}_2 + m(\dot{q}_2 + \dot{q}_1 \cos\alpha) = constant$$

- Special Cases - there are a bunch we could try! Here are a few.

  1. Wedge is Flat: $\alpha \to 0$

     Expectation: Both accelerations are zero - boring.

     Check:

$$\ddot{q}_1 = \frac{g\sin(0)(m+M)}{M + m(1-\cos^2(0))} = 0\checkmark$$

$$\ddot{q}_2 = \frac{mg\sin(0)\cos(0)}{M + m(1-\cos^2(0))} = 0\checkmark$$

  2. Wedge is vertical: $\alpha \to \pi/2$

     Expectation: Box is in freefall, wedge doesn't move.

     Check:

$$\ddot{q}_1 = \frac{g\sin(\pi/2)(m+M)}{M + m(1-\cos^2(\pi/2))} = g\checkmark$$

$$\ddot{q}_2 = \frac{mg\sin(\pi/2)\cos(\pi/2)}{M + m(1-\cos^2(\pi/2))} = 0\checkmark$$

  3. Wedge is much heavier than the box: $M >> m$

     Expectation: Box sliding down an inclined plane. Wedge doesn't move, box has constant acceleration $g\sin\alpha$.

     Check:

$$\ddot{q}_1 = \frac{g\sin\alpha(\cancelto{0}{m} + M)}{M + m\cancelto{0}{(1-\cos^2\alpha)}} = g\sin\alpha \checkmark$$

$$\ddot{q}_2 = \frac{\cancelto{0}{mg}\sin\alpha\cos\alpha}{M + m\cancelto{0}{(1-\cos^2\alpha)}} = 0 \checkmark$$

4. Box is much heavier than wedge: $m >> M$
   Expectation: Wedge will shoot out from under the box really fast. Box will essentially fall straight down.
   Check:

$$\ddot{q}_1 = \frac{g\sin\alpha(m + \cancelto{0}{M})}{\cancelto{0}{M} + m(1-\cos^2\alpha)} = \frac{g}{\sin\alpha}??$$

$$\ddot{q}_2 = \frac{mg\sin\alpha\cos\alpha}{\cancelto{0}{M} + m(1-\cos^2\alpha)} = \frac{g}{\tan\alpha}??$$

   This is a tough one to interpret. It really helps to convert the $q's$ into Cartesian coordinates:

$$\begin{aligned} \ddot{y} &= \ddot{q}_1\cos\alpha \\ &= g \end{aligned}$$

   As for $\ddot{q}_2$, this depend on alpha. As $\alpha \to \pi/2$, then the acceleration go to zero. This makes sense because the normal force from the box on the wedge is small the steeper the wedge is. As $\alpha \to 0$, the normal force is bigger, and I see that the acceleration get really big.

Now, find the time! (Don't forget the finish answering the question.)

Since the acceleration is constant, I can use my knowledge of kinematics for constant acceleration to find the time:

$$\begin{aligned} q_1(t) &= \cancelto{0}{q_1(0)} + \cancelto{0}{\dot{q}_{1,0}\,t} + \frac{1}{2}\ddot{q}_1\,t^2 \\ \to t &= \sqrt{\frac{2q_1}{\ddot{q}_1}} \\ t &= \sqrt{\frac{2d[M + m(1-\cos^2\alpha)]}{g\sin\alpha(m+M)}} \end{aligned}$$

Some sensemaking about the time:

- Length of incline: d increases

  Expectation: The time will increase.

  Check: $t \propto \sqrt{d}$✓

- Compare to fixed incline:

  Expectation: Time will be shorter for movable wedge: $\frac{t_{movable}}{t_{fixed}} < 1$.

  Check: For fixed wedge, $a = g\sin\alpha \rightarrow t_{fixed} = \sqrt{\frac{2d}{g\sin\alpha}}$

$$\frac{t_{movable}}{t_{fixed}} = \sqrt{\frac{\frac{\cancel{2d}[M+m(1-\cos^2\alpha)]}{\cancel{g\sin\alpha}(m+M)}}{\frac{\cancel{2d}}{\cancel{g\sin\alpha}}}}$$
$$= \sqrt{\frac{M+m(1-\cos^2\alpha)}{m+M}}$$
$$\leq 1 \checkmark \text{ because } (1-\cos^2\alpha) \leq 1$$