1. Imagine a spin-1/2 system prepared in the state:

$$|\psi\rangle = \sqrt{\frac{2}{5}}\,|+\rangle + i\sqrt{\frac{3}{5}}\,|-\rangle$$

Use a completeness relation to write this state in the S_x basis.

Solution

$$\begin{aligned}
|\psi\rangle &= (1)\Big(\sqrt{\frac{2}{5}}\,|+\rangle + i\sqrt{\frac{3}{5}}\,|-\rangle\Big) \\
&= \Big(|+\rangle_{x\,x}\langle +| + |-\rangle_{x\,x}\langle -|\Big)\Big(\sqrt{\frac{2}{5}}\,|+\rangle + i\sqrt{\frac{3}{5}}\,|-\rangle\Big) \\
&= \sqrt{\frac{2}{5}}\,|+\rangle_x \cancelto{\frac{1}{\sqrt{2}}}{\langle +|+\rangle} + i\sqrt{\frac{3}{5}}\,|+\rangle_x \cancelto{\frac{1}{\sqrt{2}}}{\langle +|-\rangle} \\
&+ \sqrt{\frac{2}{5}}\,|-\rangle_x \cancelto{\frac{1}{\sqrt{2}}}{\langle -|+\rangle} + i\sqrt{\frac{3}{5}}\,|-\rangle_x \cancelto{-\frac{1}{\sqrt{2}}}{\langle -|-\rangle} \\
&= \sqrt{\frac{1}{5}}\,|+\rangle_x + i\sqrt{\frac{3}{10}}\,|+\rangle_x + \sqrt{\frac{1}{5}}\,|-\rangle_x - i\sqrt{\frac{3}{10}}\,|-\rangle_x \\
&= \left(\sqrt{\frac{1}{5}} + i\sqrt{\frac{3}{10}}\right)|+\rangle_x + \left(\sqrt{\frac{1}{5}} - i\sqrt{\frac{3}{10}}\right)|-\rangle_x
\end{aligned}$$

2. Use a completeness relation to write $|-\rangle$ in the S_y basis.

Solution

$$\begin{aligned}
|-\rangle &= (1)\,|-\rangle \\
&= \Big(|+\rangle_{y\,y}\langle +| + |-\rangle_{y\,y}\langle -|\Big)\,|-\rangle \\
&= |+\rangle_y \cancelto{\frac{-i}{\sqrt{2}}}{\langle +|-\rangle} + |-\rangle_y \cancelto{\frac{i}{\sqrt{2}}}{\langle -|-\rangle} \\
&= \frac{-i}{\sqrt{2}}\,|+\rangle_y + \frac{i}{\sqrt{2}}\,|+\rangle_y \\
&= -i\left(\frac{1}{\sqrt{2}}\,|+\rangle_y - \frac{1}{\sqrt{2}}\,|+\rangle_y\right)
\end{aligned}$$