You have a system that consists of two identical (fair) six-sided dice. Imagine that you will perform an experiment where you roll the pair of dice together and record the observable: the norm of the difference between the values displayed by the two dice.

1. What are the possible results of the observable for each roll?

   Solution The possibilities are: 0,1,2,3,4,5

2. What is the theoretical probability of measuring each of those results? Assume the results are fair.

   Plot a probability histogram. Use your histogram to make a guess about where the average value is and the standard deviation.

   Solution With fair dice, there are 36 possible combinations of faces, with each combination equally probable. I can determine the norm differences by making a table of all of the possibilities:

$$
\begin{aligned}
\mathcal{P}(0) &= 6/36 \\
\mathcal{P}(1) &= 10/36 \\
\mathcal{P}(2) &= 8/36 \\
\mathcal{P}(3) &= 6/36 \\
\mathcal{P}(4) &= 4/36 \\
\mathcal{P}(5) &= 2/36
\end{aligned}
$$

3. Use your theoretical probabilities to determine a theoretical average value of the observable (*the expectation value*)? Indicate the expectation value on your histogram.

   Solution The expectation value is:

$$
\begin{aligned}
\langle D \rangle &= \sum_{n=0}^{5} \mathcal{D}_n \; P(D_n) \\
&= (0)(6/36) + (1)(10/36) + (2)(8/36) + \\
&\quad (3)(6/36) + (4)(4/36) + (5)(2/36) \\
&= 70/36 \qquad\qquad \approx 1.9
\end{aligned}
$$

4. Use your theoretical probabilities to determine the standard deviation (the *uncertainty*) of the distribution of possible results. Indicate the uncertainty on your histogram.

   Solution The uncertainty is:

$$
\begin{aligned}
\Delta D &= \sqrt{\langle D^2\rangle - \langle D\rangle^2} \\
\langle D^2\rangle &= \sum_{n=0}^{5} \mathcal{D}_n^2 \, P(D_n) \\
&= (0^2)(6/36) + (1^2)(10/36) + (2^2)(8/36) + \\
&\quad (3^2)(6/36) + (4^2)(4/36) + (5^2)(2/36) \\
&= 210/36 \\
\Delta D &= \sqrt{210/36 - (70/36)^2} \\
&\approx 1.4
\end{aligned}
$$

5. Challenge: Use
   a) Dirac bra-ket notation
   b) matrices

   to represent:

   - the possible states of the dice after a measurement is made;

     Solution

$$
|0\rangle \doteq \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \quad |1\rangle \doteq \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \quad |2\rangle \doteq \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}
$$

$$
|3\rangle \doteq \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix} \quad |4\rangle \doteq \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix} \quad |0\rangle \doteq \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}
$$

   - the state of the dice when you're shaking them up in your hand;

     Solution

$$
|\psi\rangle \doteq \begin{bmatrix}6/36\\10/36\\8/36\\6/35\\4/36\\2/36\end{bmatrix}
$$

   - an operator that represents the norm of the difference of the dice.

Solution

$$\hat{D} \doteq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

where:

$$\begin{aligned} \hat{D}\left|0\right\rangle &= 0\left|0\right\rangle \\ \hat{D}\left|1\right\rangle &= 1\left|1\right\rangle \\ \hat{D}\left|2\right\rangle &= 2\left|2\right\rangle \\ \hat{D}\left|3\right\rangle &= 3\left|3\right\rangle \\ \hat{D}\left|4\right\rangle &= 4\left|4\right\rangle \\ \hat{D}\left|5\right\rangle &= 5\left|5\right\rangle \end{aligned}$$