The following are 2 different representations for the **same** state on a quantum ring

$$|\Phi\rangle = \sqrt{\frac{1}{2}}\,|2\rangle - \sqrt{\frac{1}{4}}\,|0\rangle + i\sqrt{\frac{1}{4}}\,|-2\rangle \tag{1}$$

$$\Phi(\phi) \doteq \sqrt{\frac{1}{8\pi r_0}}\left(\sqrt{2}e^{i2\phi} - 1 + ie^{-i2\phi}\right) \tag{2}$$

1. Write down the matrix representation for the same state.

Solution

$$|\Phi\rangle \doteq \begin{pmatrix} \vdots \\ \sqrt{\frac{1}{2}} \\ 0 \\ -\sqrt{\frac{1}{4}} \\ 0 \\ i\sqrt{\frac{1}{4}} \\ \vdots \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow m=0 \\ \\ \\ \\ \end{matrix} \tag{3}$$

2. With all 3 representations, calculate the probability that a measurement of L_z will yield $0\hbar$, $-2\hbar$, $2\hbar$.

Solution The quick way to do this calculation in every representation is to read off the coefficient of the state that has $m\hbar$ as its L_z-eigenvalue and take the square of the norm. The two things to be careful about are:

a) Make sure to take the COMPLEX-norm squared of the coefficient $|c_m|^2$, not the ordinary square of the coefficient c_m^2. A quick check that you have done the right thing is to see if your answer is real and non-negative. For example,

$$\mathcal{P}_{-2\hbar} = \left|i\sqrt{\frac{1}{4}}\right|^2 = \frac{1}{4}$$

b) For the wave-function representation, it is necessary to distinguish between the part of the number in front of $e^{im\phi}$ that is normalization constant and the part that is probability amplitude. Only the probability amplitude should be used to calculate the probability. For example, the coefficient of $e^{-i2\phi}$ is

$$\sqrt{\frac{i}{8\pi r_0}} = \underbrace{\sqrt{\frac{1}{2\pi r_0}}}_{\text{normalization constant}} \qquad \overbrace{\sqrt{\frac{i}{4}}}^{\text{probability amplitude}}$$

You should also know how to do these calculations the long way. In some representations, it is not possible to do the shorthand calculation. Here is an example of each calculation:

a) Ket Representation:

$$\mathcal{P}_{-2\hbar} = |\langle -2|\Phi\rangle|^2 \tag{4}$$

$$= \left|\langle -2|\left(\sqrt{\frac{1}{2}}\,|2\rangle - \sqrt{\frac{1}{4}}\,|0\rangle + i\sqrt{\frac{1}{4}}\,|-2\rangle\right)\right|^2 \tag{5}$$

$$= \left|i\sqrt{\frac{1}{4}}\right|^2 \tag{6}$$

$$= \frac{1}{4} \tag{7}$$

b) Wave Function Representation: (This is the method you will need to use if you cannot easily see from the form of the wave function what the separate eigenstate are.)

$$\mathcal{P}_{-2\hbar} = \left| \int_0^{2\pi} \left(\sqrt{\frac{1}{2\pi r_0}} e^{-i2\phi} \right)^* \right. \tag{8}$$

$$\left. \sqrt{\frac{1}{8\pi r_0}} \left(\sqrt{2} e^{i2\phi} - 1 + i e^{-i2\phi} \right) r_0 \, d\phi \right|^2 \tag{9}$$

$$= \left| \int_0^{2\pi} \frac{1}{4\pi \cancel{r_0}} \left(\sqrt{2} e^{i4\phi} - e^{i2\phi} + i \right) \cancel{r_0} \, d\phi \right|^2 \tag{10}$$

$$= \left| \frac{1}{4\pi} (0 + 0 + 2\pi i) \right|^2 \tag{11}$$

$$= \frac{1}{4} \tag{12}$$

c) Matrix Representation:

$$\mathcal{P}_{-2\hbar} = \left| \begin{pmatrix} \ldots & 0 & 0 & \overset{\overset{m=0}{\downarrow}}{0} & 0 & 1 & \ldots \end{pmatrix}^* \begin{pmatrix} \vdots \\ \sqrt{\frac{1}{2}} \\ 0 \\ -\sqrt{\frac{1}{4}} \\ 0 \\ i\sqrt{\frac{1}{4}} \\ \vdots \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow m=0 \\ \\ \\ \\ \end{matrix} \right|^2 \tag{13}$$

$$= \frac{1}{4} \tag{14}$$

3. If you measured the z-component of angular momentum to be $2\hbar$, what would the state of the particle be immediately after the measurement is made?

Solution It will be in the $|2\rangle$ state.

4. What is the probability that a measurement of energy, E, will yield $0\frac{\hbar^2}{I}$?,$2\frac{\hbar^2}{I}$?,$4\frac{\hbar^2}{I}$?

5. If you measured the energy of the state to be $2\frac{\hbar^2}{I}$, what would the state of the particle be immediately after the measurement is made?