1 Fourier Transform of a Gaussian

Consider the Gaussian wave function $f(x) = Ne^{-x^2/2\sigma^2}$.

(a) Find the normalization constant N. Write a sentence describing the physical meaning of normalizing. (The identity $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$ may prove helpful.)

(b) Find the Fourier transform of $f(x)$ by hand. You will need to "complete the square". Sense-making: Discuss how changing the constant σ changes the shape of both $f(x)$ and its Fourier transform.

(c) Show that the Fourier transform of $f(x)$ is also normalized. (This is true for any function and is known as Parseval's identity.) Write a sentence describing the physical meaning of normalizing in this case.

2 Fourier Transform of Cosine and Sine

(a) Find the Fourier transforms of $f(x) = \cos kx$ and $g(x) = \sin kx$.

(b) Find the Fourier transform of $g(x)$ using the formula for the Fourier transform of a derivative and your result for the Fourier transform of $f(x)$. Compare with your previous answer.

(c) In quantum mechanics, the Fourier transform is the set of coefficients in the expansion of a quantum state in terms of plane waves, i.e. the function $\tilde{f}(k)$ is a continuous histogram of how much each functions e^{ikx} contributes to the quantum state. What does the Fourier transform of the function $\cos kx$ tell you about which plane waves make up this quantum state? Write a sentence or two about how this makes sense.

3 Fourier Transform of a Triangle

Consider a quantum mechanical wave packet shaped like a triangle:

$$f(x) = \begin{cases} \sqrt{\frac{3}{2\epsilon^3}}\,(x+\epsilon), & -\epsilon < x < 0 \\ -\sqrt{\frac{3}{2\epsilon^3}}\,(x-\epsilon), & 0 < x < \epsilon \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that the wave packet is normalized.

(b) For three different values of ϵ, plot both the wave packet. (All three plots should be on the same axes.)

(c) Find the Fourier Transform of the wave packet by hand. You may use technology of your choice to evaluate integrals, but do not use any built in Fourier Transform packages.

(d) For three different values of ϵ, plot both the Fourier Transform of the wave packet. (All three plots should be on the same axes.)

(e) As you change the value of ϵ so that the packet gets narrower and taller, what happens to the shape of the Fourier Transform?

(f) Show that the Fourier transform is also (norm-squared) normalized.