Finding derivatives from discrete information
You should be able to compute a partial derivative from data in a table or on a contour map. The important point is that a partial derivative is determined by the values of state variables at two "nearby" states: State 1 and state 2. For

$$\left(\frac{\partial f}{\partial g}\right)_h$$

it is essential to choose the two states so that h is constant, i.e. has the same value, in both states. Then you find the values of f and g in the two states and take the ratio of the small changes in these values:

$$\left(\frac{\partial f}{\partial g}\right)_h = \left(\frac{\triangle f}{\triangle g}\right)_h = \left(\frac{f(2)-f(1)}{g(2)-g(1)}\right)_h$$

What counts as "nearby"? You want the states close enough together that the relationship between the variables in essentially linear, but not so close that when you take the difference between two numbers that are close together you are only getting the error in the measurements. Use good professional judgement and be prepared to explain briefly why you made the choices that you did.

It is always a good idea to plot data from a table to get a sense of the uncertainty in the data: is the data approximately linear? smooth? concave-up or down? are you likely over or underestimating the derivative?

If you are reading the data off a graph, then sometimes (but not always!) the two states lie along a curve on the graph and this derivative will be the slope of the tangent line.

Zapping with d
This content is review from previous courses, but you will need to use this strategy in problem-solving. You should be able to zap a complicated algebraic expression with d using a combination of the following rules:

- zapping common functions with d, especially exponentials and logarithms.
- product rule
- ordinary chain rule

See Tevian's three short videos in the Media Gallery (10 minutes total!): Rules for Differentials, Product Rule, Chain Rule.

Physical meaning of differentials
You should be able to describe the geometry of a differentials expression as describing the relationship between small changes of various physical/geometric variables. For example:

$$df = \left(\frac{\partial f}{\partial g}\right)_h dg + \left(\frac{\partial f}{\partial h}\right)_g dh \tag{1}$$

tells us that, if g is changing but not h, then small change in f is the rate of change ("slope") $\left(\frac{\partial f}{\partial g}\right)_h$ times the small change in g.

The multivariable chain rule
You should be able to write the multivariable chain rule relevant to any state variable f in terms of any

other two (or more, depending on the number of independent variables in a system) state variables g and h.

$$df = \left(\frac{\partial f}{\partial g}\right)_h dg + \left(\frac{\partial f}{\partial h}\right)_g dh \tag{2}$$

The partial derivatives in this chain rule can be found by zapping an equation of state with d.

Realize that, in thermodynamics, there is nothing special about the state variable f. You could as easily have written the multivariable chain rule above for dg:

$$dg = \left(\frac{\partial g}{\partial f}\right)_h df + \left(\frac{\partial g}{\partial h}\right)_f dh \tag{3}$$

Manipulating differentials

The multivariable chain rule (and also any equation obtained by zapping with d) are always linear in the differentials, so it is easy to do (linear) algebra on these equations. Here is a list of allowable algebraic manipulations:

- subtract a term from both sides of an equation, which effectively moves a term from one side of an equation to the other, with an appropriate minus sign
- divide both sides of an equation by a partial derivative by turning it upside-down (and keeping the same variable constant)
- substitute one differentials expression into another

For example: in equation (2) , you can solve for dg. This example illustrates the typical allowable moves:

$$df = \left(\frac{\partial f}{\partial g}\right)_h dg + \left(\frac{\partial f}{\partial h}\right)_g dh \tag{4}$$

$$\left(\frac{\partial f}{\partial g}\right)_h dg = df - \left(\frac{\partial f}{\partial h}\right)_g dh \tag{5}$$

$$dg = \frac{1}{\left(\frac{\partial f}{\partial g}\right)_h}\left[df - \left(\frac{\partial f}{\partial h}\right)_g dh\right] \tag{6}$$

$$\Rightarrow dg = \left(\frac{\partial g}{\partial f}\right)_h df - \left(\frac{\partial g}{\partial f}\right)_h \left(\frac{\partial f}{\partial h}\right)_g dh \tag{7}$$

Interpreting partial derivatives

If you have two different differentials expressions involving the SAME differentials, then you can equate the coefficients of the differentials in the two equations.

For example, comparing equations (3) and (6) , we find:

$$\left(\frac{\partial g}{\partial f}\right)_h = \frac{1}{\left(\frac{\partial f}{\partial g}\right)_h}$$

and

$$\left(\frac{\partial g}{\partial h}\right)_f = -\frac{\left(\frac{\partial f}{\partial h}\right)_g}{\left(\frac{\partial f}{\partial g}\right)_h}$$

The most important example of "interpreting" partial derivatives in this way is from comparing the multivariable chain rule for the internal energy $U(S, V)$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

with the thermodynamic identity

$$dU = T\,dS - p\,dV$$

which gives formulas for temperature T and pressure p in terms of partials derivatives of the internal energy.

$$T = \left(\frac{\partial U}{\partial S}\right)_V \tag{8}$$

$$-p = \left(\frac{\partial U}{\partial V}\right)_S \tag{9}$$

You can use these formulas as the *definition* of T and p. For example, they tell you that the temperature describes how hard it is to change the internal energy when you change the entropy by a little bit. It's harder when the temperature is higher. Can you make a similar interpretation for the pressure?

The PDM

The PDM is a mechanical physical system that has two ways of getting energy into and out of the system (doing work on each of the two sides). The mathematics of the PDM is analogous to the mathematics of a thermodynamic system for which you can change the internal energy by doing work or by heating. So, there are analogues to all the thermodynamics you have done that also apply to the PDM. For example:

"The First Law" $dU = đW_L + đW_R$

"The Thermodynamic Identity" $dU = F_L\,dx_L + F_R\,dx_R$