An ice cream cone is to be dipped in chocolate. The cone can be described by the equation $z^2 = 9\,(x^2 + y^2)$, with $0 \leq z \leq 9$ and x, y, and z in centimeters. The dipping process is such that the resulting (surface) density of chocolate on the cone is given by $\sigma = 1 - \frac{z}{9}$ in grams per square centimeter. Find the total amount of chocolate on the cone.

(There is no ice cream on the cone!)

Missing /var/www/paradigms_media_2/media/activity_media/cone.png

Solution The subtlety in this problem is that the slant height of the cone is *not* the same as the vertical height. Using *dvecr* ensures that the area element is correct!

We need to calculate $dA = |d\vec{A}| = |d\vec{r}_1 \times d\vec{r}_2|$, starting from the cylindrical expression

$$d\vec{r} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

It is easy to see that $d\vec{r}_1 = s\,d\phi\,\hat{\phi}$, since s and z are constant as you go around the cone. However, along the slanted edge of the cone, both of z and s are changing, so $d\vec{r}_2 = ds\,\hat{s} + dz\,\hat{z}$. In order to express this last expression in terms of a single variable, we must relate z and s, which can be done using proportional reasoning.

Drawing similar triangles vertically, we can conclude that

$$\frac{R}{H} = \frac{s}{z}$$

where the radius $R = 3$ and height $H = 9$ are given. In other words, we have $z = 3s$, and therefore $dz = 3\,ds$.

We can use these expressions to eliminate either s or z. Choosing the latter, we obtain

$$dA = |s\,d\phi\,\hat{\phi} \times (\hat{s} + 3\,\hat{z})\,ds| = |\hat{s} + 3\,\hat{z}|\,s\,ds\,d\phi = \sqrt{10}\,s\,ds\,d\phi$$

We can now evaluate the desired integral, namely

$$\int \sigma\,dA = \int_0^{2\pi} \int_0^3 \left(1 - \frac{s}{3}\right) \sqrt{10}\,s\,ds\,d\phi = 20\sqrt{10}\,\pi \left(\frac{s^2}{2} - \frac{s^3}{9}\right)\Bigg|_0^3$$

yielding a final answer of $3\pi\sqrt{10}$ grams.