Find the Fourier transform of a plane wave.

Solution Your first job is to figure out what is meant by the words "plane wave." In this case, we are probably thinking about a one-dimensional wave, so that we can do a one-dimensional Fourier transform. Call the spatial direction x, so the wave is

$$e^{i(kx-\omega t)} \tag{1}$$

where the dispersion relation (i.e. the relationship $\omega(k)$ between ω and k) will depend on the particular wave equation you are considering. While we are doing the Fourier transform, we only care about the x dependence of the function. The time dependence will just "go along for the ride" as an overall phase, so I will simplify the calculation by choosing $t = 0$. Leave the time dependence in if you want to see how that works out.

Now, be very careful! We have already used the variable named k in the definition of our plane wave, so we need to pick a DIFFERENT variable name k' to put into the formula for the Fourier transform. It doesn't matter which k you put the prime on, but once you have chosen, make sure you are consistent throughout the rest of whatever larger calculation you are doing, including its interpretation.

$$\begin{aligned}
\mathcal{F}\left(e^{ikx}\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik'x}\, e^{ikx}\, dx && (2)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx && (3)\\
&= \sqrt{2\pi}\, \delta(k-k') && (4)
\end{aligned}$$

In the last line, we have used the exponential definition of the delta function (as a function of k, not as a function of x). Read The Exponential Representation of the Dirac Delta Function for a derivation of this representation.