Consider a quantum state for a free particle with mass m that, at $t = 0$, is a superposition of three different energy eigenstates: $\psi_{p_0}(x)$, $\psi_{p_0+\delta p}(x)$, and $\psi_{p_0-\delta p}(x)$. Let the probabilities be $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, respectively.

1. Write down the state in wavefunction at $t = 0$. (don't do any simplification yet).

Solution

$$\psi(x,0) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0}{\hbar}x} + \frac{1}{2}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0+\delta p}{\hbar}x} + \frac{1}{2}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0-\delta p}{\hbar}x}$$

2. What is the shape of this wavefunction? Use your favorite plotting software to graph the real and imaginary parts of this wavefunction. To graph the function, you'll have to pick a numerical value for p_0 and δp.

Solution I'll choose $\frac{p_0}{\hbar} = 2$ and $\delta p = 0.1$ and I won't worry about the constant out from except the probability amplitudes:

$$Re(\psi) = \frac{1}{\sqrt{2}}\cos 2x + \frac{1}{2}\cos 2.1x + \frac{1}{2}\cos 1.9x$$
$$Im(\psi) = 0.7\sin 2x + 0.5\sin 2.1x + 0.5\sin 1.9x$$

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3. At this stage, try to reorganize/simplify your wavefunction by factoring out $\psi_{p_0}(x)$.

Solution

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0}{\hbar}x}\left(\frac{1}{\sqrt{2}} + \frac{1}{2}e^{i\frac{\delta p}{\hbar}x} + \frac{1}{2}e^{-i\frac{\delta p}{\hbar}x}\right)$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0}{\hbar}x}\left[\frac{1}{\sqrt{2}} + \cos\left(\frac{\delta p x}{\hbar}\right)\right]$$

4. What will the state be at a later time t? (don't do any simplification yet).

Solution Starting with the energy eigenstate expansion, add a time-dependent phase to each term, using the time evolution formula:

$$\psi(x,t) = \sum_m \phi_m(x)e^{-i\frac{E_m}{\hbar}t}$$

So we have:

$$\begin{aligned}\psi(x,t) =& \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0}{\hbar}x}e^{-i\frac{p_0^2}{2m\hbar}t} \\ &+ \frac{1}{2}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0+\delta p}{\hbar}x}e^{-i\frac{(p_0+\delta p)^2}{2m\hbar}t} \\ &+ \frac{1}{2}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0-\delta p}{\hbar}x}e^{-i\frac{(p_0-\delta p)^2}{2m\hbar}t}\end{aligned}$$

5. At this stage, try to reorganize your wavefunction by factoring out $\psi_{p_0}(x,t)$.

Solution

$$\begin{aligned}\psi(x,t) =& \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0}{\hbar}x}e^{-i\frac{p_0^2}{2m\hbar}t} \\ &\left[\frac{1}{\sqrt{2}} + \frac{1}{2}e^{i\frac{\delta p}{\hbar}x}e^{-i\frac{(2p_0\delta p+\delta p^2)}{2m\hbar}t}\right. \\ &\left. + e^{-i\frac{\delta p}{\hbar}x}e^{-i\frac{(2p_0\delta p-\delta p^2)}{2m\hbar}t}\right]\end{aligned}$$

Here we let all factors of δp of power greater than 1 (in red) go to 0.

$$\begin{aligned}\psi(x,t) =& \frac{1}{2}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0}{\hbar}(x-\frac{p_0}{2m}t)} \\ &\left(\frac{1}{\sqrt{2}} + \frac{1}{2}e^{i\frac{\delta p}{\hbar}(x-\frac{p_0}{m}t)} + \frac{1}{2}e^{-i\frac{\delta p}{\hbar}(x+i\frac{p_0}{m}t)}\right)\end{aligned}$$

Now we use Euler's identity for Cosine: $\cos(kx) = \frac{e^{ikx}+e^{-ikx}}{2}$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p_0}{\hbar}(x-\frac{p_0}{2m}t)}\left[\frac{1}{\sqrt{2}} + \cos\left(\frac{\delta p}{\hbar}\left[x-\frac{p_0}{m}t\right]\right)\right]$$

The velocity in blue is the velocity of the **carrier** wave and the velocity in red is the velocity of the **envelope**. Note: the velocity of the envelope matches the velocity I would expect for a classical particle.