Use power series methods to find the general solution of

$$\left(\frac{d^2}{dx^2} - A\right) y(x) = 0 \tag{1}$$

around the point $x = 0$.

Solution Ansatz: Assume a power series solution. Since we are asked to expand around the point $x_0 = 0$, we choose powers of $x - x_0 = x$, i.e. let

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots && (2)\\ \frac{d}{dx} y &= 0 + c_1 x^0 + 2c_2 x^1 + 3c_3 x^2 + \dots && (3)\\ \frac{d^2}{dx^2} y &= 0 + 0 \quad + 2c_2 x^0 + 3 \cdot 2c_3 x^1 + \dots && (4)\\ & && (5) \end{aligned}$$

Plug these expressions into the differential equation.

$$\begin{aligned} 0 = &\qquad\qquad +2c_2 + 3 \cdot 2c_3 x + \dots && (6)\\ &-A(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) && (7)\\ & && (8) \end{aligned}$$

The first row in this expression comes from the second derivative term and the second row comes from the term proportional to y. Now, the goal is to add together all the terms with the same power of x. To do this, it is necessary to shift one row with respect to the other.

$$\begin{aligned} 0 = &\qquad \Longleftarrow +2c_2 + 3 \cdot 2c_3 x + \dots && (9)\\ &- A(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) && (10)\\ &= 2c_2 + 3 \cdot 2c_3 x + \dots && (11)\\ &- Ac_0 + Ac_1 x + Ac_2 x^2 + Ac_3 x^3 + \dots && (12)\\ &= (2c_2 + Ac_0) + (3 \cdot 2c_3 + Ac_1) x + \dots && (13) \end{aligned}$$

Now utter the magic words: "Since this equation must be equal to zero for *all* values of x, the coefficients of each power of x must separately be equal to zero." Thus, we obtain:

$$2c_2 + Ac_0 = 0 \Rightarrow c_2 = \frac{A}{2} c_0 \tag{14}$$

$$3 \cdot 2c_3 - Ac_1 = 0 \Rightarrow c_3 = \frac{A}{3 \cdot 2} c_1, \text{ etc.} \tag{15}$$

Solution "Dummy" index method (Use expressions in terms of an infinite sum):

Ansatz: Assume a power series solution. Since we are asked to expand around the point $x_0 = 0$, we choose powers of $x - x_0 = x$, i.e. let

$$y = \sum_{m=0}^{\infty} c_m x^m \tag{16}$$

$$\frac{d}{dx} y = \sum_{m=0}^{\infty} c_m \, m \, x^{m-1} \tag{17}$$

$$\frac{d^2}{dx^2} y = \sum_{m=0}^{\infty} c_m \, m(m-1) \, x^{m-2} \tag{18}$$

$$\tag{19}$$

Plug these expressions into the differential equation.

Shift the "dummy" index on some of the sums so that each sum contains the same power of x.

$$0 = \underbrace{\sum_{m=0}^{\infty} c_m \, m(m-1) \, x^{m-2}}_{\text{let } m \to m+2} - A \sum_{m=0}^{\infty} c_m x^m \tag{20}$$

$$= \sum_{m+2=0}^{\infty} c_{m+2} \, (m+2)(m+1) \, x^m - A \sum_{m=0}^{\infty} c_m x^m \tag{21}$$

$$= c_0 \cancelto{0}{(2-2)} (2-1) + c_1 (1-0) \cancelto{0}{(1-1)} + \sum_{m=0}^{\infty} c_{m+2} \, (m+2)(m+1) \, x^m - A \sum_{m=0}^{\infty} c_m x^m \tag{22}$$

$$= \sum_{m=0}^{\infty} \left[c_{m+2} \, (m+2)(m+1) - A \, c_m \right] x^m \tag{23}$$

Now utter the magic words: "Since this equation must be equal to zero for *all* values of x, the coefficients of each power of x must separately be equal to zero," i.e. the recurrence relation is given by:

$$c_{m+2} = \frac{A}{(m+2)(m+1)} \, c_m \tag{24}$$

Solution Using the recurrence relation:

Now, by plugging in successive values of m, we can *recursively* find values for the unknown coefficients c_m. For example, when we plug in $m = 0$, we obtain

$$c_2 = \frac{A}{2} \, c_0.$$

However, when we plug in $m = 2$, we obtain

$$c_4 = \frac{A}{4 \cdot 3} c_2 \tag{25}$$
$$= \frac{A}{4 \cdot 3} \frac{A}{2} c_0 \tag{26}$$

Recursively means that we just found c_4 in terms of c_2, but we can then use the previous result to write c_4 in terms of c_0.

Notice that I did NOT multiply out the factors in the denominator. I do this so that I have a better chance of identifying a pattern in the coefficients. It is common to get factorials in the denominator, as you can see emerging in this case.

Also notice that we started with a second order differential equation, so we expect two linearly independent solutions, each multiplied by an overall constant. There is nothing in our solution so far that tells us the value of c_0 or c_1, so these two constants will become the overall constants for the solution.

Also notice that the recurrence relation (24) contains only c_{m+2} and c_m and does not contain c_{m+1}, i.e. it skips a step. This behavior is common in applications, but by no means always true! When this does happen, if you are expanding around $x = 0$, then the solutions will be pure even or pure odd functions of x. The physical situation should also reflect this symmetry.

Solution The coefficients:

The first few coefficients that we get are:

$$c_0 = c_0 \tag{27}$$
$$c_2 = \frac{A}{2 \cdot 1} c_0 \tag{28}$$
$$c_4 = \frac{A^2}{4 \cdot 3 \cdot 2 \cdot 1} c_0 \tag{29}$$
$$c_6 = \frac{A^3}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} c_0 \tag{30}$$
$$\tag{31}$$
$$c_1 = c_1 \tag{32}$$
$$c_3 = \frac{A}{3 \cdot 2 \cdot 1} c_1 \tag{33}$$
$$c_5 = \frac{A^2}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} c_1 \tag{34}$$
$$c_7 = \frac{A^3}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} c_1 \tag{35}$$
$$\tag{36}$$

Leading to the approximate solutions:

$$y(x) = c_0 \left(1 + \frac{A}{2 \cdot 1} x^2 + \frac{A^2}{4 \cdot 3 \cdot 2 \cdot 1} x^4 + \frac{A^3}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} x^6 + \dots \right) \tag{37}$$

$$+ c_1 \left(1x + \frac{A}{3 \cdot 2 \cdot 1} x^3 + \frac{A^2}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} x^5 + \frac{A^3}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} x^7 + \dots \right) \tag{38}$$

$$= c_0 \cosh(\sqrt{A} x) + \frac{c_1}{\sqrt{A}} \sinh(\sqrt{A} x) \tag{39}$$

In this particularly simple case, you may be able to recognize the series for hyperbolic sine and cosine. More typically, you will not be able to "sum the series," i.e. recognize the series in terms of known functions. If you were able to recognize the series, likely you would have been able to solve the problem using simpler methods than power series solutions.

Note that if A is a positive real number, we could rearrange the two solutions above to get two exponential solutions (while losing information about which solution is even or odd). If A is a negative real number, we could factor the minus sign out of the square root as $\pm i$ and rearrange the two solutions above to get sines and cosines.