In previous work, we reduced the central force problem to a pair of uncoupled ordinary differential equations for the variables r and ϕ as functions of time, i.e.

$$\mu r^2 \dot{\phi} = \ell = \text{constant} \tag{1}$$

and

$$\ddot{r} = \frac{\ell^2}{\mu^2 r^3} + \frac{1}{\mu} f(r) \tag{2}$$

If we are only interested in the shape of the orbit, we can do something simpler than solving for $r(t)$ and $\phi(t)$. Instead of using the variable t as a parameter in (1) and (2), we will use the variable ϕ and solve for $r(\phi)$, the polar equation for the shape of the orbit.

To do this, we need to change the time derivatives into ϕ derivatives.

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} \tag{3}$$

It turns out that the differential equation which we obtain will be much easier to solve if we also change independent variables from r to

$$u = r^{-1} \tag{4}$$

(I don't know how to motivate this clever guess.) Therefore,

$$\frac{dr}{dt} = \frac{\ell}{\mu r^2}\frac{dr}{d\phi} \tag{5}$$

$$= -\frac{\ell}{\mu}\frac{d\,r^{-1}}{d\phi} \tag{6}$$

$$= -\frac{\ell}{\mu}\frac{du}{d\phi} \tag{7}$$

(To verify the middle equality, work from right to left.) Then the second derivative is given by

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\frac{dr}{dt} = \frac{\ell}{\mu}\,u^2\,\frac{d}{d\phi}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2}{\mu^2}\,u^2\,\frac{d^2u}{d\phi^2} \tag{8}$$

Plugging (4) and (8) into (2), dividing through by u^2, and rearranging, we obtain the orbit equation

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{\ell^2}\,\frac{1}{u^2}\,f\left(\frac{1}{u}\right) \tag{9}$$

For the special case of inverse square forces $f(r) = -k/r^2$ (spherical gravitational and electric sources), it turns out that the right-hand side of (9) is constant so that the equation is particularly easy to solve. First solve the homogeneous equation (with $f(r) = 0$), which is just the harmonic oscillator equation with general solution

$$u_{\mathrm{h}} = A\cos(\phi - \delta) \tag{10}$$

Add to this any particular solution of the inhomogeneous equation (with $f(r) = -k/r^2$). By inspection, such a solution is just the constant

$$u_{\mathrm{p}} = \frac{\mu\,k}{\ell^2} \tag{11}$$

so that the general solution of (9) for an inverse square force is

$$r^{-1} = u = u_{\rm h} + u_{\rm p} = A\cos(\phi - \delta) + \frac{\mu\, k}{\ell^2} \tag{12}$$

Then solving for r in (12) we obtain

$$r = \frac{1}{\frac{\mu k}{\ell^2} + A\cos(\phi - \delta)} = \frac{\frac{\ell^2}{\mu k}}{1 + A'\cos(\phi - \delta)} \tag{13}$$

This is the equation for a conic section, expressed in polar coordinates.

It turns out that the choice of the constant δ just determines the orientation of the orbit in space. If all we care about is the shape, we can choose $\delta = 0$.