Find a third order approximation to the electrostatic potential $V(\vec{r})$ for one of the following physical situations.

1. Two charges $+Q$ and $+Q$ are placed on a line at $z = D$ and $z = -D$ respectively.

   a) On the x-axis for $|x| \ll D$?

   b) On the z-axis for $|z| \ll D$?

   c) On the x-axis for $|x| \gg D$?

   d) On the z-axis for $|z| \gg D$?

2. Two charges $+Q$ and $-Q$ are placed on a line at $z = +D$ and $z = -D$ respectively.

   a) On the x-axis for $|x| \ll D$?

   b) On the z-axis for $|z| \ll D$?

   c) On the x-axis for $|x| \gg D$?

   d) On the z-axis for and $|z| \gg D$?

Work out your problem by brainstorming together on a big whiteboard and also answer the following questions:

- For what values of $\vec{r}$ does your series converge?
- For what values of $\vec{r}$ is your approximation a good one?
- Which direction would a test charge move under the influence of this electric potential?

If your group gets done early, go on to another problem. The fourth problem in each set is the most challenging, and the most interesting.

Solution

Solution 1 First of all, recall the expression for the potential due to a point charge, namely $V = \frac{kq}{r}$. What is r? The distance from the charge to the point under consideration. So "r" in this expression is standing for $|\vec{r} - \vec{r}'|$, where $\vec{r}'$ is the location of the charge, and $\vec{r}$ the point where the potential is being evaluated.

Consider case 1d, with charges of Q located at $(0, 0, \pm D)$, being evaluated on the z-axis far from the origin. So $\vec{r}' = \pm D\hat{z}$ for the two charges, and $\vec{r} = z\hat{z}$, where $|z| \gg D$. The combined potential for this configuration is obtained by superimposing (i.e. adding) the separate potentials for each charge, resulting in

$$V(z) = \frac{kQ}{|z - D|} + \frac{kQ}{|z + D|} \tag{1}$$

where the absolute value signs are necessary because z can be positive or negative. For $z > 0$, we can drop the absolute value signs, so we consider that case first. The answer for $z < 0$ will follow immediately, since both terms change sign, so all we will have to do is add a global minus sign.

Combining terms (and assuming $z > D > 0$) leads to

$$V(z) = kQ\left(\frac{1}{z-D} + \frac{1}{z+D}\right) = \frac{2kQz}{z^2 - D^2} \tag{2}$$

In order to expand $V(z)$ in terms of a power series, we must first identify a small, dimensionless parameter, thus answering the question, "power series with respect to what." In this case, since $z \gg D$, the desired parameter is $\frac{D}{z}$, so we need to rewrite $V(z)$ by factoring z^2 out of the denominator, yielding

$$V(z) = \frac{2kQ}{z}\frac{1}{1 - \frac{D^2}{z^2}} = \frac{2kQ}{z}\left(1 - \frac{D^2}{z^2}\right)^{-1} \tag{3}$$

Finally, using the binomial expansion, we obtain

$$V(z) = \frac{2kQ}{z}\left(1 + \frac{D^2}{z^2} + \frac{D^4}{z^4} + ...\right) \tag{4}$$

To obtain an expression valid for both signs of z, recall that for $z < -D < 0$, the absolute value in the original expression for $V(z)$ contributes a global minus sign *and* $z < 0$, so $V(z)$ remains positive (as it must, by symmetry). So our final answer can be written

$$V(z) = \frac{2kQ}{|z|}\left(1 + \frac{D^2}{z^2} + \frac{D^4}{z^4} + ...\right) \tag{5}$$

The remaining cases can be handled similarly.

Solution

Solution 2 Instead of considering the cases with $z > 0$ and $z < 0$ separately, we can instead handle both cases at once. Starting from

$$V(z) = \frac{kQ}{|z-D|} + \frac{kQ}{|z+D|} \tag{6}$$

we can factor out $|z|$ from the denominator in both terms to obtain

$$V(z) = \frac{kQ}{|z|}\left(\frac{1}{1 - \frac{D}{z}} + \frac{1}{1 + \frac{D}{z}}\right) \tag{7}$$

Why aren't there absolute value signs on the terms in parentheses? Because $D \ll z$, so $\frac{D}{z} \ll 1$, which ensures that the denominators are always positive. These terms can now be added together, as in the previous solution, yielding the same final result, with the factor $|z|$ already in place.

Alternatively, the two terms can be expanded separately, since

$$\frac{1}{1 \mp \frac{D}{z}} = 1 \pm \frac{D}{z} + \frac{D^2}{z^2} \pm \frac{D^3}{z^3} + \frac{D^4}{z^4} + ... \tag{8}$$

> for $\frac{D}{z} \ll 1$. Adding these two power series together term by term now yields
>
> $$V(z) = \frac{2kQ}{|z|}\left(1 + \frac{D^2}{z^2} + \frac{D^4}{z^4} + ...\right) \tag{9}$$
>
> as before.