Consider a parabola $y = \alpha x^2$ for 0 m $\leq x \leq$ 5 m. Find the total charge if the charge density on the parabola is proportional to the width of the parabola and varies from 0 Coulombs per meter at $x = 0$ m to 11 Coulombs per meter at $x = 5$ m.

Solution This problem illustrates the "use what you know" strategy.

First we need to find a formula for the charge density, λ. The words "proportional to the width" mean that the charge density varies linearly with respect to x since x is a parameter that measures (half of) the width of the parabola in the xy-plane. Therefore we know that $\lambda(x) = \alpha x + \beta$. We can use the conditions that $\lambda(0\text{ m}) = 0$ C/m and $\lambda(5\text{ m}) = 11$ C/m to find the values of α and β.

$$\begin{aligned}
0\text{ C/m} &= \lambda(0\text{ m}) = \alpha(0\text{ m}) + \beta && (1)\\
&\Rightarrow \beta = 0\text{ C/m} && (2)\\
11\text{ C/m} &= \lambda(5\text{ m}) = \alpha 5\text{ m} + 0\text{ C/m} && (3)\\
&\Rightarrow \alpha = \frac{11}{5}\text{ C/m}^2 && (4)\\
\rightarrow \lambda(x) &= \left(\frac{11}{5}\text{ C/m}^2\right)x && (5)
\end{aligned}$$

Next, we need to find $d\ell = |d\vec{r}|$ along the parabola. Start with the expression for $d\vec{r}$ on the plane in rectangular coordinates

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

and then plug in what you know about the equation of a parabola and its differential:

$$\begin{aligned}
y &= 3x^2 && (6)\\
\Rightarrow dy &= 6x\,dx && (7)\\
\Rightarrow d\vec{r} &= dx\,\hat{x} + 6x\,dx\,\hat{y} && (8)\\
&= dx(\hat{x} + 6x\,\hat{y}) && (9)
\end{aligned}$$

Now find the magnitude of $d\vec{r}$

$$\begin{aligned}
d\ell = |d\vec{r}| &= \sqrt{dx(\hat{x} + 6x\,\hat{y}) \cdot dx(\hat{x} + 6x\,\hat{y})} && (10)\\
&= \sqrt{dx^2(1 + 36x^2)} && (11)\\
&= \sqrt{1 + 36x^2}\,|dx| && (12)
\end{aligned}$$

Plug all of these results into the formula for the total charge of a one-dimensional charge distribution.

$$\begin{aligned}
Q_{\text{total}} &= \int \lambda d\ell && (13)\\
&= \int_0^5 \frac{11}{5} x \sqrt{1 + 36x^2}\,|dx| && (14)
\end{aligned}$$

We can remove the absolute values signs from $|dx|$ because we are integrating from small values of x to large values of x so dx is always positive. The integral can be performed with a simple $u = 1 + 36x^2$.