A helix with 17 turns has height H and radius R. Charge is distributed on the helix so that the charge density increases like (i.e. proportional to) the square of the distance up the helix. At the bottom of the helix the linear charge density is $0\ \frac{\mathrm{C}}{\mathrm{m}}$. At the top of the helix, the linear charge density is $13\ \frac{\mathrm{C}}{\mathrm{m}}$. What is the total charge on the helix?

Solution Let's start by finding the charge density on the helix. We know that the charge density varies like the square of the distance up the helix. You might want to know whether "distance" means the vertical height z or the distance along the helix $ds = Rd\phi$, but since these are proportional to each other, it doesn't matter. I will choose to work with z, but if you work with ds or even $d\phi$, you will get the right answer.

$$\lambda(z) = \alpha z^2 \tag{1}$$
$$13 = \alpha H^2 \tag{2}$$
$$\Rightarrow \alpha = \frac{13}{H^2} \tag{3}$$
$$\lambda(z) = \frac{13}{H^2}\, z^2 \frac{\mathrm{C}}{\mathrm{m}} \tag{4}$$

We will chop up the helix into little pieces and find the charge $\lambda\, d\ell$ on each little piece. (Note that properly, this $\lambda(z)$ has the correct dimensions, since z/H is unitless—I will suppress the $\frac{\mathrm{C}}{\mathrm{m}}$ and restore it at the very end.) Then we will add up (integrate) the charge from each of the little pieces.

So the next thing we need to find is $d\ell$ on a little piece of the helix.

$$d\ell = |d\vec{r}| \tag{5}$$
$$= |ds\,\hat{s} + sd\phi\,\hat{\phi} + dz\,\hat{z}| \tag{6}$$
$$= |R\,d\phi\,\hat{\phi} + dz\,\hat{z}| \tag{7}$$
$$\tag{8}$$

We are trying to do a line integral, so we need to find ds in terms of a single parameter, but both ϕ and z are changing. We need to know the relationship between ϕ and z to proceed further.

$$\phi = \beta z \tag{9}$$
$$34\pi = \beta H \tag{10}$$
$$\Rightarrow \beta = \frac{34\pi}{H} \tag{11}$$
$$\phi = \frac{34\pi}{H}\, z \tag{12}$$
$$\Rightarrow d\phi = \frac{34\pi}{H}\, dz \tag{13}$$

Therefore,

$$\begin{aligned} d\ell &= |d\vec{r}| && (14)\\ &= |R\,d\phi\,\hat{\phi} + dz\,\hat{z}| && (15)\\ &= |(R\,\frac{34\pi}{H}\,\hat{\phi} + \hat{z})\,dz| && (16)\\ &= \sqrt{\left(\frac{34\pi\,R}{H}\right)^2 + 1}\,dz && (17) \end{aligned}$$

Putting all the pieces together, we have:

$$\begin{aligned} Q &= \int_0^H \lambda(z)\,d\ell && (18)\\ &= \int_0^H \frac{13}{H^2}\,z^2\sqrt{\left(\frac{34\pi\,R}{H}\right)^2 + 1}\,dz && (19)\\ &= \frac{13}{H^2}\sqrt{\left(\frac{34\pi\,R}{H}\right)^2 + 1}\left(\int_0^H z^2\,dz\right) && (20)\\ &= \frac{13}{H^2}\sqrt{\left(\frac{34\pi\,R}{H}\right)^2 + 1}\,\frac{z^3}{3}\Bigg|_0^H && (21)\\ &= \frac{13H}{3}\sqrt{\left(\frac{34\pi\,R}{H}\right)^2 + 1}\,\frac{\mathrm{C}}{\mathrm{m}} && (22) \end{aligned}$$

Sense-making: How can you be confident that an answer like this is correct? The dimensions work out (you can see that the square root is dimensionless, and the extra H cancels the remaining m to leave only Coulombs. It might be worth trying a special case just to be sure. How about as R goes to 0, $Q = \frac{13H}{3}$? This case corresponds to a line of charge (still with a quadratic density), which means that $d\ell = dz$ and matching our solution.

H goes to 0 is also tempting, corresponding to the helix becoming 17 circles piled on top of each other, but tricky because you have to be willing to change your assumption to the charge increasing with ϕ, as there is no height to increase over. Uh-oh, there's a problem in the denominator. Fortunately, you can multiply through by H! This leaves $Q = 13\frac{34\pi R}{3}$. This is indeed the result you get if you go back and recognize that $d\ell = Rd\phi$ (integrated from 0 to 34π) and

$$\lambda(\phi) = 13\frac{\phi^2}{(34\pi)^2} \qquad (23)$$

In either of the other cases, you might also decide to do a power series, either for a very wide or very tall helix, to extract some more detailed behavior near those limits.A helix with 17 turns has height H and radius R.