Find the upward pointing flux of the electric field $\vec{E} = E_0\, z\, \hat{\boldsymbol{z}}$ through the part of the surface $z = -3s^2 + 12$ (cylindrical coordinates) that sits above the (x, y)-plane.

Break this problem into steps:

- Sketch the paraboloid and the vector field.
- Sketch a representative $d\vec{\boldsymbol{A}}$ and calculate $d\vec{\boldsymbol{A}}$ algebraically.
- Calculate the flux.

Solution We want to compute $\int \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}}\, dA$ over the surface of the paraboloid given, or if you prefer (I find it even simpler to think of this way), just $\int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$. We know $\vec{\boldsymbol{E}}$, so we really just need to draw little "patches" of area $d\vec{\boldsymbol{A}}$ on the surface, and figure out what those area vectors look like. That's not so bad—just draw a little patch somewhere on the surface! Mine is a little square which is $d\vec{\boldsymbol{r}}_1$ by $d\vec{\boldsymbol{r}}_2$. Here I really need a picture, but let's try to visualize it with words:

Let $d\vec{\boldsymbol{r}}_1$ point in the $\hat{\boldsymbol{\phi}}$ direction, with length $s\, d\phi$ (Do you see why?) That's the "bottom" leg of a little patch, which sweeps around in the $\hat{\boldsymbol{\phi}}$ direction as you change ϕ just a little. It has no z component, and in fact no radial component either, because this paraboloid sweeps symmetrically around the z-axis.

The other leg has to be "tilted" up (along) the paraboloid surface. I drew myself a little picture and noticed that this vector points "up the paraboloid" (in order that our cross product will give the outward normal), so it would be written

$$d\vec{\boldsymbol{r}}_2 = ds\, \hat{\boldsymbol{s}} + dz\, \hat{\boldsymbol{z}} \tag{1}$$

(Draw it, make sure this makes sense to you!) Do **NOT** add an explicit minus sign on the radial component. The limits of integration will take care of this. Now, "use what you know." $d\vec{\boldsymbol{r}}_2$ lives on a parabola, so s and z are related by $z = -3s^2 + 12$, so $dz = -6s\, ds$. Decide whether you want to integrate with respect to s or z. I will choose s, so I need to change all the z's to s's, (including the z in the expression for the electric field!) Therefore, $d\vec{\boldsymbol{r}}_2$ becomes

$$d\vec{\boldsymbol{r}}_2 = (\hat{\boldsymbol{s}} - 6s\, \hat{\boldsymbol{z}})\, ds \tag{2}$$

Now we're all set.

$$\begin{aligned} d\vec{\boldsymbol{A}} &= d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2 \\ &= s\, d\phi\, \hat{\boldsymbol{\phi}} \times (\hat{\boldsymbol{s}} - 6s\, \hat{\boldsymbol{z}})\, ds \\ &= (-s\, \hat{\boldsymbol{z}} - 6s^2\, \hat{\boldsymbol{s}})\, ds\, d\phi \end{aligned} \tag{3}$$

We only CARE about the $\hat{\boldsymbol{z}}$ component of this expression, because we're about to dot it with $\vec{\boldsymbol{E}} = E_0\, z\, \hat{\boldsymbol{z}}$, and thus we find

$$\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = E_0(-3s^2 + 12)(-s)\, ds\, d\phi \tag{4}$$

Almost done! This is our integrand, and we still need to do the double integral. The $d\phi$ integration is sweet, we sweep around the circle and get 2π, since nothing depends on ϕ. So we just have one simple single integration to do, along ds, which should run from 2 to 0. (Oh, do you see why that is 2? This surface is like a hat, and reaches $z = 0$ at the point $s = 2$.) Make sure to start at $s = 2$, the outside of the hat, and run to $s = 0$ at the top of the hat because that is the direction that $d\vec{\boldsymbol{r}}_2$ points.

Therefore, the integral that we have to do is:

$$\begin{aligned} \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} &= \int_2^0 \int_0^{2\pi} E_0(3s^3 - 12s)\, ds\, d\phi \\ &= 24\pi\, E_0 \end{aligned} \tag{5}$$

By the way, you can check this answer using the divergence theorem, which we'll talk about later in the week! The surface integral of $\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$ must equal the volume integral of the divergence of E, (where $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = E_0$ in this case). So we just need the volume of the paraboloid. I leave that as an exercise for you (just slice it into circular "plates"); it works! One thing to watch out for is that the surface integral needs to include the flux through the BOTTOM of the paraboloid, too. (The divergence theorem only works for **closed** surfaces.) But we're ok, because $z = 0$ there, and so there is no extra flux from that surface.