Time Evolution of the Infinite Square Well

Launch the "Quantum Bound States" PhET. Pause the simulation at t=0.
Make the square well potential as deep as you can on the screen to approximate an infinite well.

1. What happens to the energy levels if you:
   a) change the width of the well?
   b) change the mass of the particle?

   How your observations are consistent with the equation for the energy eigenvalues?

   Solution We can see as we make the well longer or make the particle mass larger, we have energy lines that get closer together. Or more energy states fit into our well. This is reflected in the eigenvalue equation for the infinite square well:

   $$\hat{H} |n\rangle = E_n |n\rangle = \frac{n^2\pi^2\hbar^2}{2mL^2} |n\rangle \tag{1}$$

   From this equation we see the energy scales inversely to m and L^2 and that explains why we get lower energies for larger particles and wells.

2. At t=0, what do the energy eigenstate wavefunctions look like?
   a) The real part?
   b) The imaginary part?
   c) The magnitude?
   d) The phase?

   How do these shapes and colors make sense?

   Solution at $t = 0$ the imagniary part of the wavefunction looks to be 0 everywhere no matter the energy level I chose. Whereas the real part shows a wave that goes to 0 on both ends of the well and as a number of bumps matching the energy level. The magnitude is similar to the real part, just that it is always positive because it is $|\psi|$. The phase seems to jump back and forth between 0 and $\frac{\pi}{2}$ every half a wavelength.

3. At t=0, what does the probability density look like for the energy eigenstates?
   a) How is the shape of the probability density related to the shape of the wavefunction?
   b) For the n=2 energy eigenstate (in other words, the first excited state)
      i. If you were looking for the particle in the box, where is the particle most likely to be? Explain.
      ii. What is the expectation value of the position of the particle? Explain.

Solution The probability density is the norm sqaure of the wavefunction, $|\psi|^2$. This means that all the wavefunction pieces will be sqaured, with the real and imaginary parts being folded in together. At $t = 0$ the imaginary part of ψ is 0, so the probability desnsity looks like just the real part norm sqaured. We still have the same number of bumps as the real part of the wave function, just they're all positive this time.
For the $n = 2$ eigenstate, we have two bumps in the probability desnity halfway between the center and edges of the well, at $-\frac{L}{4}$ and $\frac{L}{4}$ if we put the origin at center of the well as is shown initially in the PhET. Therefore, we would be most likely to find the particle near those two bumps.
For the expecation value, we have a symmetric function and so we will have a expectation value of 0. This occurs because we are equally likely to find it at both bumps which are evenly spaces apart and symmetric around the origin, so when we take the average of our measurements, we will get 0. This is an interesting result since the probability of measuring the particle at the origin is 0 since the probability density is 0 there. However, expectation values are classical results which don't need to be values we can measure exactly in quantum mechanics.

4. As time passes, what do the energy eigenstates do? (What do they look like?)
   a) The real part?
   b) The imaginary part?
   c) The magnitude?
   d) The phase?

   Explain why you see what you see.

Solution We see the real and imaginary parts of the wave function fluctuate back and forth. In particular, they look to be standing waves that are $\frac{\pi}{2}$ off of each other in phase, with the imaginary blue wave following the real red one. The magnitude is time independent and doesn't change, which means the components of the real and imaginary parts are filling in for each other as they move through the wave cycle. The phase seems like it is measured as a color from the real, not the relative phase between the real and imaginary parts. It fluctuates between 0 and 2π as the wave functions go around a cycle. Looking at the 2nd excited state again, we see the different bumps in the wave are $\frac{\pi}{2}$ off of each other at any given time.

5. As time passes, what does the probability density of the energy eigenstates look like?

   Explain why you see what you see.

Solution The probability density doesn't change in time for these individual energy states. These are called stationary states because their probabilities as a whole will not depend on time. This means all the $e^{-i\frac{E_n}{\hbar}t}$'s must come out in the wash when we take the norm sqaure to get $|\psi|^2$.

6. Create a superposition state:

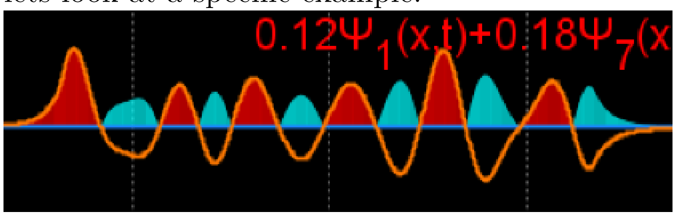
   a) What does the wavefunction look like at t=0?

   b) How does the wavefunction evolved with time?

   c) How does the probability density evolved with time?

   Explain why you see what you see.

Solution This can have a lot of different results depending on how we set up the superposition, lets look at a specific example:



We see the bumps can now be offset and not as well ordered. We still see the phase is off by $\frac{\pi}{2}$ in each defined bump. We still have a 0 imaginary wavefunction everywhere.

This all changes as we evolve in time, getting pieces of real and imagiary parts as before, while the phase varies in not easily interpretable ways since we have a lot of eigenstates combined here with different frequencies of waves.

The probability density does change with time here, still with about as many bumps as the highest eigenstate we used in the superposition. However, because the bumps move from left to right now, sometimes bumps will combine with each other and others will drop to 0 at various times and we won't be able to interpret the highest energy by looking at just the bumps anymore.