The following are 2 different representations for the **same** state on a quantum ring

$$|\Phi\rangle = \sqrt{\frac{1}{2}}\,|2\rangle - \sqrt{\frac{1}{4}}\,|0\rangle + i\sqrt{\frac{1}{4}}\,|{-2}\rangle \tag{1}$$

$$\Phi(\phi) \doteq \sqrt{\frac{1}{8\pi r_0}}\left(\sqrt{2}e^{i2\phi} - 1 + ie^{-i2\phi}\right) \tag{2}$$

1. Write down the matrix representation for the same state.

Solution

$$|\Phi\rangle \doteq \begin{pmatrix} \vdots \\ \sqrt{\frac{1}{2}} \\ 0 \\ -\sqrt{\frac{1}{4}} \\ 0 \\ i\sqrt{\frac{1}{4}} \\ \vdots \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow m=0 \\ \\ \\ \\ \end{matrix} \tag{3}$$

2. With all 3 representations, calculate the probability that a measurement of L_z will yield $0\hbar$, $-2\hbar$, $2\hbar$.

Solution The quick way to do this calculation in every representation is to read off the coefficient of the state that has $m\hbar$ as its L_z-eigenvalue and take the square of the norm. The two things to be careful about are:

a) Make sure to take the COMPLEX-norm squared of the coefficient $|c_m|^2$, not the ordinary square of the coefficient c_m^2. A quick check that you have done the right thing is to see if your answer is real and non-negative. For example,

$$\mathcal{P}_{-2\hbar} = \left|i\sqrt{\frac{1}{4}}\right|^2 = \frac{1}{4}$$

b) For the wave-function representation, it is necessary to distinguish between the part of the number in front of $e^{im\phi}$ that is normalization constant and the part that is probability amplitude. Only the probability amplitude should be used to calculate the probability. For example, the coefficient of $e^{-i2\phi}$ is

$$\sqrt{\frac{i}{8\pi r_0}} = \underbrace{\sqrt{\frac{1}{2\pi r_0}}}_{\text{normalization constant}} \qquad \overbrace{\sqrt{\frac{i}{4}}}^{\text{probability amplitude}}$$

You should also know how to do these calculations the long way. In some representations, it is not possible to do the shorthand calculation. Here is an example of each calculation:

a) Ket Representation:

$$\begin{aligned}
\mathcal{P}_{-2\hbar} &= \left|\langle -2|\Phi\rangle\right|^2 && (4)\\
&= \left|\langle -2|\left(\sqrt{\frac{1}{2}}\,|2\rangle - \sqrt{\frac{1}{4}}\,|0\rangle + i\sqrt{\frac{1}{4}}\,|-2\rangle\right)\right|^2 && (5)\\
&= \left|i\sqrt{\frac{1}{4}}\right|^2 && (6)\\
&= \frac{1}{4} && (7)
\end{aligned}$$

b) Wave Function Representation: (This is the method you will need to use if you cannot easily see from the form of the wave function what the separate eigenstate are.)

$$\begin{aligned}
\mathcal{P}_{-2\hbar} &= \left|\int_0^{2\pi}\left(\sqrt{\frac{1}{2\pi r_0}}e^{-i2\phi}\right)^*\right. && (8)\\
&\qquad \left.\sqrt{\frac{1}{8\pi r_0}}\left(\sqrt{2}e^{i2\phi} - 1 + ie^{-i2\phi}\right)\, r_0\, d\phi\right|^2 && (9)\\
&= \left|\int_0^{2\pi}\frac{1}{4\pi \cancel{r_0}}\left(\sqrt{2}e^{i4\phi} - e^{i2\phi} + i\right)\cancel{r_0}\, d\phi\right|^2 && (10)\\
&= \left|\frac{1}{4\pi}\left(0 + 0 + 2\pi i\right)\right|^2 && (11)\\
&= \frac{1}{4} && (12)
\end{aligned}$$

c) Matrix Representation:

$$\begin{aligned}
\mathcal{P}_{-2\hbar} &= \left|\begin{pmatrix}\ldots & 0 & 0 & \overset{\overset{m=0}{\downarrow}}{0} & 0 & 1 & \ldots\end{pmatrix}^* \begin{pmatrix}\vdots\\ \sqrt{\frac{1}{2}}\\ 0\\ -\sqrt{\frac{1}{4}}\\ 0\\ i\sqrt{\frac{1}{4}}\\ \vdots\end{pmatrix} \begin{matrix}\\ \\ \\ \leftarrow m=0\\ \\ \\ \\ \end{matrix}\right|^2 && (13)\\
&= \frac{1}{4} && (14)
\end{aligned}$$

For the other probabilities, with any representation, we get:

$$\mathcal{P}_{-2\hbar} = \frac{1}{4} \tag{15}$$
$$\mathcal{P}_{0\hbar} = \frac{1}{4} \tag{16}$$
$$\mathcal{P}_{2\hbar} = \frac{1}{2} \tag{17}$$

3. If you measured the z-component of angular momentum to be $2\hbar$, write down the full resultant state immediately after the measurement.

Solution It will be in the $|2\rangle$ state because that is the only eigenstate corresponding to meauring $2\hbar$, so it will be in that state with 100% probability.

4. If an energy measurement is performed on the state $\Phi(\phi)$, what is the probability that the energy measurement will yield each of the following values: $0\frac{\hbar^2}{I}$?,$2\frac{\hbar^2}{I}$?,$4\frac{\hbar^2}{I}$?

Solution I'll do this in the ket representation, but I could do it in any of them. Since our energy eigenvalyue equation on the ring goes like:

$$\hat{H}\left|m\right\rangle = E_m\left|m\right\rangle = \frac{m^2\hbar^2}{2I}\left|m\right\rangle \tag{18}$$

We see our energy eigenvalues for these measurements are:

$$0\frac{\hbar^2}{I} \rightarrow \left|0\right\rangle \tag{19}$$
$$2\frac{\hbar^2}{I} \rightarrow \frac{(2)^2\hbar^2}{2I} \rightarrow \left|2\right\rangle, \left|-2\right\rangle \tag{20}$$
$$4\frac{\hbar^2}{I} \rightarrow \frac{(\sqrt{8})^2\hbar^2}{2I} \rightarrow ??? \tag{21}$$

There is no eigenstate corresponding to $m = \sqrt{8}$, so we know $E = 4\frac{\hbar^2}{I}$ isn't a valid measurement on the ring. Therefore we know:

$$\mathcal{P}_{4\frac{\hbar^2}{I}} = 0 \tag{22}$$
$$\tag{23}$$

For $0\frac{\hbar^2}{I}$, we take the probability like we normally do, with the norm sqaure of one inner product:

$$\mathcal{P}_{0\frac{\hbar^2}{I}} = \left|\left\langle 0|\Phi\right\rangle\right|^2 = \frac{1}{4} \tag{24}$$
$$\tag{25}$$

However, for the last measurement, $E = 2\frac{\hbar^2}{I}$, we have two eigenstates which correspond to the measurement, so we need to sum the probabilities of all these eigenstates which yield the same measurement, like so:

$$\mathcal{P}_{2\frac{\hbar^2}{I}} = |\langle 2|\Phi\rangle|^2 + |\langle -2|\Phi\rangle|^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \tag{26}$$

$$\tag{27}$$

We can still check our work and make sure our probabilities all add to 1!

$$\mathcal{P}_{0\frac{\hbar^2}{I}} + \mathcal{P}_{2\frac{\hbar^2}{I}} + \mathcal{P}_{4\frac{\hbar^2}{I}} = \frac{1}{4} + \frac{3}{4} + 0 = 1 \tag{28}$$

$$\tag{29}$$

And they do! Don't forget all the old sensemaking stategies still work when you're working with degnerate states.

5. If you measured the energy of the state to be $2\frac{\hbar^2}{I}$, write down the full resultant state immediately after the measurement.

Solution Just like how we have to add the probabilties together for degenerate measurements, when we make a measurement that corresponds to two different eigenstates in our overall state, we end up with a superposition of those measurement corresponding eigenstates immediately after the measurement. We could do this formally with the projection postulate:

$$|\psi_{out}\rangle = \frac{\hat{P}\,|\psi_{in}\rangle}{\langle\psi_{in}|\,\hat{P}\,|\psi_{in}\rangle} \tag{30}$$

However, I will do it a simpler way. All we have to do is cross out the terms with eigenstates states that don't correspond to our measurement in our overall state:

$$|\Phi\rangle = \sqrt{\frac{1}{2}}\,|2\rangle - \cancel{\sqrt{\frac{1}{4}}\,|0\rangle} + i\sqrt{\frac{1}{4}}\,|-2\rangle \tag{31}$$

Then we just have to renormalize (making sure all our probabilities still add to 1!). There are a few ways to do this, but I just like adding up the norm sqaures of each coefficent and then dividing by the root of the total:

$$|c_2|^2 + |c_{-2}|^2 = \left|\sqrt{\frac{1}{2}}\right|^2 + \left|i\sqrt{\frac{1}{4}}\right|^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \tag{32}$$

So I'll divide the overall state by $\sqrt{\frac{3}{4}}$, that gives:

$$|\Phi\rangle = \sqrt{\frac{2}{3}}\,|2\rangle + i\sqrt{\frac{1}{3}}\,|-2\rangle \tag{33}$$

> Notice that the phase just came along for the ride on both kets, and the realative probabilities between the eigenstates stayed that same! By this I mean I was twice as likely to measure $|2\rangle$ compared to $|-2\rangle$ before the measurement and the state changing, and after the measurement, I am still twice as likely to measure $|2\rangle$ compared to $|-2\rangle$!