Consider the following normalized quantum state on a unit ring:

$$\Phi(\phi) = \sqrt{\frac{8}{3\pi r_0}} \sin^2(3\,\phi)\cos(\phi) \tag{1}$$

1. Translate this state into eigenfunction, bra/ket, and matrix representations. Remember that you can use any of these representations in the following calculations.

Solution First we will rewrite the function in terms of energy eigenstates. To do this we can expand our sines and cosines in terms of exponentials. N.B. It is not always possible to immediately see how a function can be expanded in terms of imaginary exponentials. In that case, you MUST figure out the expansion using the techniques of (exponential) Fourier series.

$$\begin{aligned} \Phi(\phi) &= \sqrt{\frac{8}{3\pi r_0}} \sin^2(3\phi)\cos(\phi) && (2)\\ &= -\sqrt{\frac{8}{3\pi r_0}} \left(\frac{e^{3i\phi} - e^{-3i\phi}}{2i}\right)^2 \left(\frac{e^{i\phi} + e^{-i\phi}}{2}\right) && (3)\\ &= -\sqrt{\frac{1}{12}}\sqrt{\frac{1}{2\pi r_0}} \left(e^{7i\phi} - 2e^{i\phi} + e^{-5i\phi} + e^{5i\phi} - 2e^{-i\phi} + e^{-7i\phi}\right) && (4) \end{aligned}$$

We can now translate this into bra/ket notation by remembering that $|m\rangle = \sqrt{\frac{1}{2\pi r_0}} e^{im\phi}$.

$$|\Phi\rangle = -\sqrt{\frac{1}{12}}\,|7\rangle - \sqrt{\frac{1}{12}}\,|5\rangle + \sqrt{\frac{4}{12}}\,|1\rangle + \sqrt{\frac{4}{12}}\,|{-1}\rangle - \sqrt{\frac{1}{12}}\,|{-5}\rangle - \sqrt{\frac{1}{12}}\,|{-7}\rangle \tag{5}$$

and into matrix notation

$$\Phi \doteq \begin{pmatrix} \vdots \\ -\sqrt{\frac{1}{12}} \\ 0 \\ -\sqrt{\frac{1}{12}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{4}{12}} \\ 0 \\ \sqrt{\frac{4}{12}} \\ 0 \\ 0 \\ 0 \\ -\sqrt{\frac{1}{12}} \\ 0 \\ -\sqrt{\frac{1}{12}} \\ \vdots \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow m = 0 \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \tag{6}$$

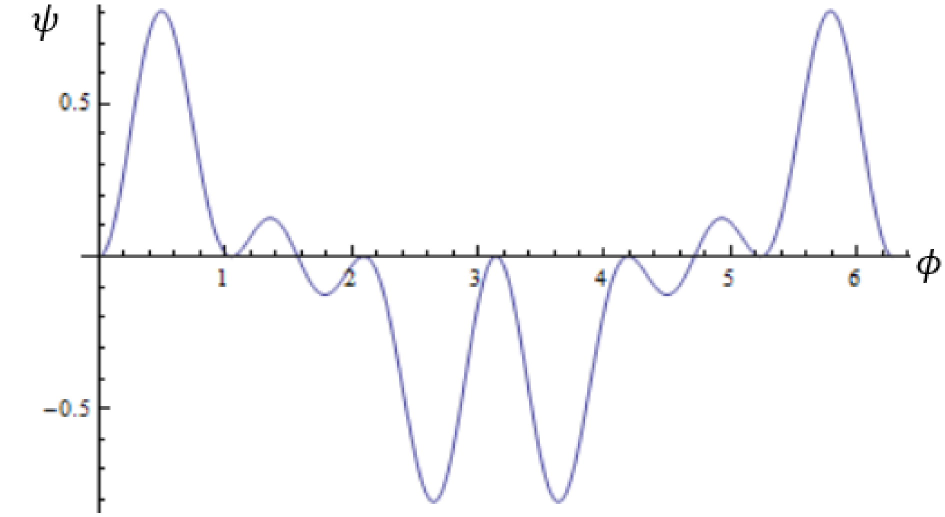
2. What is the expectation value of L_z in this state?

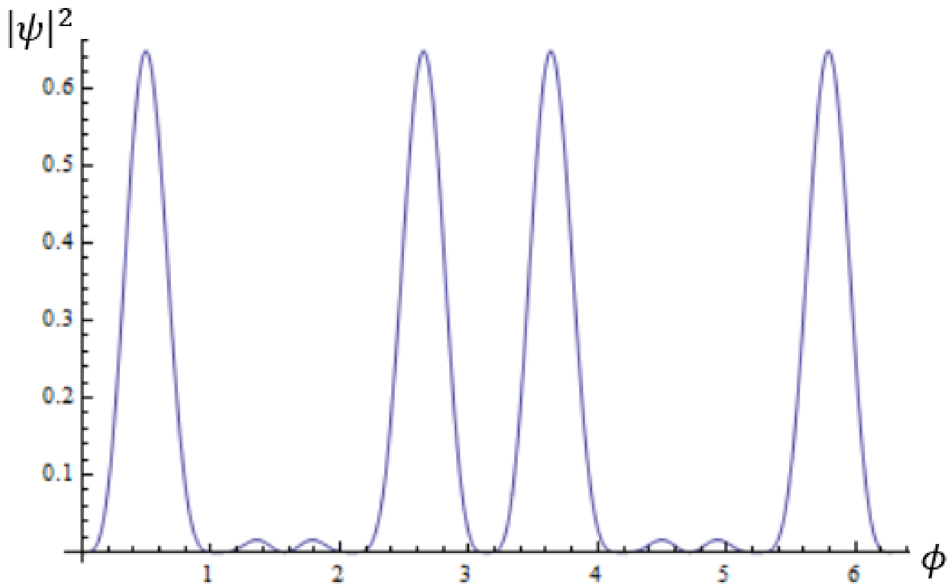
Solution The form of the answer should be a number with units matching the quantity of interest. In this case, we want an average value of L_z given a large number of measurements on systems identical to what is given.
The expectation value here is zero. For every state with positive m that appears in the wave function, there is a corresponding state with negative m (and thus negative eigenvalue of L_z) that appears in the wave function with an identical probability. Thus, the weighted average over them all will be zero.

$$\langle L_z \rangle = \sum_{m=-\infty}^{\infty} \mathcal{P}_{(m\hbar)} m\hbar \tag{7}$$

$$= 0 \tag{8}$$

3. The wave function and it's probability density are plotted below. (I have set $r_0 = 1$ to make the plotting easier). What features of these graphs (if any) tell you the expectation value of L_z in this state?





Solution We can see from the graph of the probability amplitude that (at $t = 0$) this state is real and an EVEN function of ϕ. A real, even function of ϕ means that if we substitute $-\phi$ for ϕ, we get the same function. But, this substitution also swaps solutions with positive and negative values of m which corresponds to swapping solutions with postive and negative values of the $z-$ component of angular momentum. So the expectation value of the $z-$ component of angular momentum must be zero.

Even if the probability amplitude were not an even function, we still have that the probability density is an even function. This fact is sufficient to argue that the probabilities of each of the basis states with opposit signs of m are the same in the expression for the expectation value

$$\langle L_z \rangle = \sum_{m=-\infty}^{\infty} \mathcal{P}_{(m\hbar)} m\hbar \tag{9}$$

$$= 0 \tag{10}$$

4. What is the probability that the particle can be found in the region $0 < \phi < \frac{\pi}{4}$? Repeat your calculation in the region $\frac{\pi}{4} < \phi < \frac{3\pi}{4}$?

Solution *Mathematica* has been used to evaluate the integrals.

$$\mathcal{P}(0 < \phi < \pi/4) = \int_0^{\pi/4} \left| \sqrt{\frac{8}{3\pi r_0}} \sin^2(3\,\phi)\cos(\phi) \right|^2 r_o d\phi \tag{11}$$

$$\int_0^{\pi/4} \frac{8}{3\pi r_0} \sin^4(3\phi)\cos^2(\phi)\, r_0 d\phi \tag{12}$$

$$\approx 0.24. \tag{13}$$

$$\mathcal{P}(\pi/4 < \phi < 3\pi/4) = \int_{\pi/4}^{3\pi/4} \left| \sqrt{\frac{8}{3\pi r_0}} \sin^2(3\,\phi)\cos(\phi) \right|^2 r_0 d\phi \tag{14}$$

$$= \int_{\pi/4}^{3\pi/4} \frac{8}{3\pi r_0} \sin^4(3\phi)\cos^2(\phi)\, r_0 d\phi \tag{15}$$

$$\approx 0.019. \tag{16}$$

Sensemaking: Notice that the graph of the probability density is overall larger in the first region than in the second region, so we expect the probability to be higher there. We also expect that the total probability will be one. There are four "bumps" where the probability density is large, so we expect the total probability to be about $4 \times 0.24 < 1$.