Use the power series method to solve Legendre's Equation

$$\frac{d^2P}{dz^2} - \frac{2z}{1-z^2}\frac{dP}{dz} - \frac{A}{1-z^2}P = 0 \tag{1}$$

Solution Assume that the solution can be written as a Taylor series

$$P(z) = \sum_{n=0}^{\infty} a_n\, z^n \tag{2}$$

and solve for the coefficients a_n. Then we have

$$\frac{dP}{dz} = \sum_{n=0}^{\infty} a_n\, n\, z^{n-1} \tag{3}$$

$$\frac{d^2P}{dz^2} = \sum_{n=0}^{\infty} a_n\, n(n-1)\, z^{n-2} \tag{4}$$

Multiply (1) by $1-z^2$ to clear the denominators from the differential equation and then plug in (2)-(4) to obtain

$$0 = \sum_{n=2}^{\infty} a_n\, n(n-1)\, z^{n-2} - z^2 \sum_{n=0}^{\infty} a_n\, n(n-1)\, z^{n-2} \tag{5}$$

$$- 2z \sum_{n=0}^{\infty} a_n\, n\, z^{n-1} - A \sum_{n=0}^{\infty} a_n\, z^n \tag{6}$$

We are free to choose the lower limit in the first sum to be 2 instead of 0 since the $n = 0, 1$ terms are zero because of the factor of $n(n-1)$.

In (6), the summation variable n is a dummy variable (just like a dummy variable of integration). Therefore, in the first sum, we can shift $n \to n+2$. *Pay special attention to what this does to the lower limit of the sum.* At the same time, bring any overall factors of z into the corresponding sums. Finally, since each sum now has a factor of z^n and runs over the same range, group the sums together.

$$0 = \sum_{n=0}^{\infty} a_{n+2}\, (n+2)(n+1)\, z^n - \sum_{n=0}^{\infty} a_n\, n(n-1)\, z^n$$

$$- 2 \sum_{n=0}^{\infty} a_n\, n\, z^n - A \sum_{n=0}^{\infty} a_n\, z^n \tag{7}$$

$$= \sum_{n=0}^{\infty} \left[a_{n+2}\, (n+2)(n+1) - a_n\, n(n-1) - 2\, a_n\, n - A\, a_n\right] z^n \tag{8}$$

Now comes the MAGIC part. Since (8) is true *for all values of z*, the coefficient of each term in the sum (i.e. the expression in the square brackets) must be equal to zero for each separate value of n, i.e.

$$a_{n+2}\, (n+2)(n+1) - a_n\, n(n-1) - 2\, a_n\, n - A\, a_n = 0 \tag{9}$$

and therefore we can solve for a_{n+2} in terms of a_n

$$a_{n+2} = \frac{n(n+1)+A}{(n+2)(n+1)}\, a_n \tag{10}$$

The recurrence relation (10) allows us to find a_2, a_4, etc. in terms of the arbitrary constant a_0 and also to find a_3, a_5, etc. in terms of the arbitrary constant a_1. Thus, for the second order differential equation (1) we get two arbitrary coefficients, as expected.

In general, the solutions of an ordinary linear differential equation can blow-up only where the coefficients of the equation itself are singular, in this case at $z = \pm 1$, which correspond to the north and south poles $\theta = 0, \pi$. But there is nothing special about physics at these points, only the choice of coordinates is special there. Therefore, we want to choose solutions of (1) which are regular (non-infinite) at $z = \pm 1$. This is an important example of a problem where the choice of coordinates for a partial differential equation end up imposing boundary conditions on the ordinary differential equation which comes from it. But polynomials cannot blow-up on the interval $-1 \leq z \leq 1$. So if we choose the special values for the separation constant A to be $A = -\ell(\ell+1)$ where ℓ is a non-negative integer, we see from (10) that for $n \geq \ell$ the coefficients become zero and the series terminates. The solutions for these special values of A are polynomials of degree ℓ, denoted P_ℓ, and called Legendre polynomials.

Notice that the differential equation

$$\frac{d^2P}{dz^2} - \frac{2z}{1-z^2}\frac{dP}{dz} + \frac{\ell(\ell+1)}{1-z^2}P = 0 \tag{11}$$

is a **different** equation for different values of ℓ. For a given value of ℓ, you should expect two solutions of (11). Why? We have only given one. It turns out that the "other" solution for each value of ℓ is not regular (i.e. it blows up) at $z = \pm 1$. In cases where the separation constant A does not have the special value $l(l+1)$, it turns out that *both* solutions blow up. We discard these irregular solutions as unphysical for the problem we are solving.