Consider the following state for a quantum particle confined to the surface of the unit sphere.

$$\psi(\theta,\phi) = \sqrt{\frac{2}{9}}Y_2^{-2}(\theta,\phi) - \sqrt{\frac{4}{9}}Y_5^{-2}(\theta,\phi) + i\sqrt{\frac{3}{9}}Y_5^{3}(\theta,\phi) \tag{1}$$

1. Write this state in Dirac notation.

Solution The spherical harmonics $Y_\ell^m(\theta,\phi)$ are normalized. (Hurray!) So the bra/ket notation and the wave function notation use the same probability amplitudes (coefficients). Therefore,

$$\psi(\theta,\phi) = \sqrt{\frac{2}{9}}Y_2^{-2}(\theta,\phi) - \sqrt{\frac{4}{9}}Y_5^{-2}(\theta,\phi) + i\sqrt{\frac{3}{9}}Y_5^{3}(\theta,\phi) \tag{2}$$

$$\rightarrow \tag{3}$$

$$|\psi\rangle = \sqrt{\frac{2}{9}}\,|2,-2\rangle - \sqrt{\frac{4}{9}}\,|5,-2\rangle + i\sqrt{\frac{3}{9}}\,|5,3\rangle \tag{4}$$

The only trick here is to remember that the subscript on the spherical harmonic corresponds to the quantum number ℓ and the superscript corresponds to the quantum number m, i.e. $|\ell,m\rangle \doteq Y_\ell^m(\theta,\phi)$

2. Write out the energy eigenvalues for particle on a sphere from E_0 to E_5.

Solution The energy eigenvalues for the sphere are $E_\ell = -\frac{\hbar^2}{2I}\,\ell(\ell+1)$, so we have

$$E_0 = 0 \tag{5}$$

$$E_1 = -2\,\frac{\hbar^2}{2I} \tag{6}$$

$$E_2 = -6\,\frac{\hbar^2}{2I} \tag{7}$$

$$E_3 = -12\,\frac{\hbar^2}{2I} \tag{8}$$

$$E_4 = -20\,\frac{\hbar^2}{2I} \tag{9}$$

$$E_5 = -30\,\frac{\hbar^2}{2I} \tag{10}$$

I chose not to cancel the two in the denominator with the numerical factors in the numerator so that you can see and begin to learn to recognize how the numbers $\ell(\ell+1)$ change.

3. What is the probability that a measurement of the z-component of angular momentum will return a result of $-2\hbar$?

Solution The operator for the z-component of angular momentum $\hat{L}_z$ is degenerate, i.e. it has more than one eigenstate with the same eigenvalue. Any eigenstate with $m = -2$ will contribute to the requested probability. So, for each such eigenstate, project the given state onto the eigenstate to find the probability amplitude. Then take the square of the norm to find the probability. Finally, add the probabilities.

$$\mathcal{P}(m=-2) = \sum_{\ell=0}^{\infty} |\langle \ell, -2|\psi\rangle|^2 \tag{11}$$

$$= \sum_{\ell=0}^{\infty} \left| \langle \ell, -2| \left(\sqrt{\frac{2}{9}}\,|2,-2\rangle - \sqrt{\frac{4}{9}}\,|5,-2\rangle + i\sqrt{\frac{3}{9}}\,|5,3\rangle \right) \right|^2 \tag{12}$$

$$= \left| \sqrt{\frac{2}{9}} \right|^2 + \left| \sqrt{-\frac{4}{9}} \right|^2 \tag{13}$$

$$= \frac{6}{9} \tag{14}$$

Technically, you can't get any $m = -2$ states until ℓ is at least 2, so you might want to start your sum at $\ell = -2$. Since starting at $\ell = 0$ won't give any false positives, I don't usually worry about that nuance.

4. What is the probability that an energy measurement will return a result of E_5?

Solution The Hamiltonian operator $\hat{H}$ for the energy is degenerate, i.e. it has more than one eigenstate with the same eigenvalue. Any eigenstate with $\ell = 5$ will contribute to the requested probability. So, for each such eigenstate, project the given state onto the eigenstate to find the probability amplitude. Then take the square of the norm to find the probability. Finally, add the probabilities.

$$\mathcal{P}(\ell=5) = \sum_{m=-5}^{5} |\langle 5, m|\psi\rangle|^2 \tag{15}$$

$$= \sum_{m=-5}^{5} \left| \langle 5, m| \left(\sqrt{\frac{2}{9}}\,|2,-2\rangle - \sqrt{\frac{4}{9}}\,|5,-2\rangle + i\sqrt{\frac{3}{9}}\,|5,3\rangle \right) \right|^2 \tag{16}$$

$$= \left| -\sqrt{\frac{4}{9}} \right|^2 + \left| i\sqrt{\frac{3}{9}} \right|^2 \tag{17}$$

$$= \frac{7}{9} \tag{18}$$

For $\ell = 5$, we can only have m-values between -5 and 5, so notice the limits on the sums.