Consider the state of a particle of mass μ confined to a unit sphere (a rigid rotor)

$$|\psi_1\rangle = -\frac{1}{\sqrt{2}}|2,1\rangle + \frac{1}{\sqrt{2}}|2,-1\rangle$$

Whenever appropriate, state explicitly the quantum postulate(s) that justify your predictions and calculations for each question.

1. If you measure the z-component of the angular momentum, what are the possible values you could obtain with nonzero probability? What are the probabilities for these measurements?

Solution We could obtain $L_z = \hbar, -\hbar$, this is because L_z goes with the quantum number m and we see $m = 1$ and -1 represented in the kets in our given state.
We can just read off the coefficents of the kets and take the norm sqaure of them to get the probabilities:

$$P(L_z = \hbar) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$P(L_z = -\hbar) = \left|+\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

2. If you measure the square of the angular momentum, what are the possible values you could obtain with nonzero probability? What are the probabilities for these measurements?

Solution Here we can only measure $L^2 = 6\hbar^2$ since ℓ is the quantum number we care about and we only see $\ell = 2$ represented in our state. Therefore, we know we will always get $L^2 = 6\hbar^2$, but we could also add two probabilities from each ket together.

3. If you measure the energy, what are the possible values you could obtain with nonzero probability? What are the probabilities for these measurements?

Solution Remember $H = \frac{L^2}{2I} = \frac{\ell(\ell+1)\hbar^2}{2I}$, so we end up with the same probabilties as we get for L^2 on the sphere. We only have kets giving energy measurements of $E_2 = \frac{6\hbar^2}{2I}$, and simialrly, we get it with 100% probability.

4. Calculate the expectation values of the observables for $\hat{L}_z$, $\hat{L}^2$, and $\hat{H}$.

Solution For H and $\hat{L}^2$, we only have 1 measurement, so we know:

$$\langle\hat{H}\rangle = \frac{6\hbar^2}{2I}$$

$$\left\langle\hat{L}^2\right\rangle = 6\hbar^2$$

For $\hat{L}_Z$, we need to sum the probabiltities times the measurements:

$$\langle\hat{L}_z\rangle = \sum_m P_m(m\hbar) = \frac{1}{2}(\hbar) + \frac{1}{2}(-\hbar) = 0$$

5. What is the wavefunction for the same particle at an arbitrary time t?

> **Solution** Since our state is already expressed in terms of eigenstates of the Hamiltonian of the Sphere, we can just apply the time evolution formula (A.K.A. add the complex exponentials next to each ket):
>
> $$|\psi_1(t)\rangle = -\frac{1}{\sqrt{2}}e^{-i6\hbar t}|2,1\rangle + \frac{1}{\sqrt{2}}e^{-i6\hbar t}|2,-1\rangle$$
>
> Since all kets have the same energy, we can recognize these complex coefficents represent an overall phase change as time evolves.

6. How will your answers to the questions 2-5 change with time?

> **Solution** None of them will change since our time evolution is the same for both kets, we end up with an overall phase difference not a relative one, and because of that, our probabilities will not change with time.

Repeat all of the questions above for the state:

$$|\psi_2\rangle = \frac{\sqrt{3}}{2}e^{\frac{i\pi}{3}}|2,1\rangle + \frac{1}{2}|3,1\rangle$$

> **Solution** The non-zero probabilities and measures they corespond to for each are:
>
> $$P(L_z = \hbar) = 1$$
>
> $$P(L^2 = 6\hbar^2) = \frac{3}{4}$$
>
> $$P(L^2 = 12\hbar^2) = \frac{1}{4}$$
>
> $$P\left(H = \frac{6\hbar^2}{2I}\right) = \frac{1}{4}$$
>
> $$P\left(H = \frac{12\hbar^2}{2I}\right) = \frac{1}{4}$$
>
> Our expectation values are:
>
> $$\langle \hat{L}_z \rangle = \hbar$$
>
> $$\langle \hat{L}^2 \rangle = \frac{3}{4}(6\hbar^2) + \frac{1}{4}(12\hbar^2) = \frac{15}{2}\hbar^2$$
>
> $$\langle \hat{H} \rangle = \frac{3}{4}\left(\frac{6\hbar^2}{2I}\right) + \frac{1}{4}\left(\frac{12\hbar^2}{2I}\right) = \frac{15}{4I}\hbar^2$$
>
> Our time evolution is:
>
> $$|\psi_2(t)\rangle = -\frac{\sqrt{3}}{\sqrt{2}}e^{-i6\hbar t}e^{\frac{i\pi}{3}}|2,1\rangle + \frac{1}{\sqrt{2}}e^{-i12\hbar t}|3,1\rangle$$

Here we have relative phase difference as we envolve in time (because the complex exponentials with t depend on different energies/frequencies). Therefore, some of our measurements MIGHT depend on time.

However, since all of our measurements above corresponded to operators which commute with the Hamiltonian (the operator which determines how states evolve in time), we won't see any time dependence in the measurements we made above. However, we could see time dependent probabilties for operators like position or momentum which DON'T commute with the Hamiltonian on the sphere.