Find an expression for angular momentum in polar coordinates.

Solution We know that an expression for velocity in polar coordinates is:

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}$$

Then angular momentum is

$$\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} && (1)\\
&= \vec{r} \times \mu\vec{v} && (2)\\
&= r\hat{r} \times \mu(\dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}) && (3)\\
&= r\hat{r} \times \mu(\cancel{\dot{r}\hat{r}} + r\dot{\phi}\,\hat{\phi}) && (4)\\
&= \mu r^2\dot{\phi}(\hat{r} \times \hat{\phi}) && (5)\\
&= \begin{cases} \mu r^2\dot{\phi}\,\hat{z} & \text{cylindrical coordinates} \\ -\mu r^2\dot{\phi}\,\hat{\theta} & \text{spherical coordinates} \end{cases} && (6)
\end{aligned}$$

Recall that in the equatorial plane, the basis vector $\hat{\theta}$ points *downward*.